Differential Operator Notation

1 The differential operator

Let

$$D^i := \frac{d^i}{dt^i}$$

be short for an operator to take the ith derivative. Note that

$$D^0 := \frac{d^0}{dt^0}$$

denotes taking no derivative.

Then we can write the non-homogeneous equation as

$$L(x(t)) = g(t) \text{ with } L = D^n + b_{n-1}(t)D^{n-1} + \ldots + b_1(t)D + b_0(t)D^0$$

where the action is to "apply each D operator" term to $x(t)$ and where

$$b_i(t)D^i$$

means to take the "ith" derivative and then multiply by the function b_i. Thus

$$b_0(t)D^0$$

simply means to multiply the function b_0 with $x(t)$ (take the "zero" derivative first).

Definition 1. This operator is called a **linear operator**, because

$$L(x(t) + y(t)) = L(x(t)) + L(y(t))$$

$$L(cx(t)) = cL(x(t)), \quad c \text{ is real}$$

Note: If $x_1(t), \ldots, x_k(t)$ are solutions of $L(x(t)) = 0$ and if $c_1, \ldots, c_k$ are any real constants, then, since L is a linear operator,

$$x(t) = c_1x_1(t) + c_2x_2(t) + \ldots + c_kx_k(t)$$

is also a solution of $L(x(t)) = 0$.

Example 1. Thus we can write

$$\text{(a)} \quad x''(t) = 3\frac{b(t)}{t-2}x'(t) - e^{t^2-2}x(t)$$

as

$$L(x(t)) = 0 \quad \text{with } L = D^2 - 3\frac{b(t)}{t-2}D - e^{t^2-2}D^0.$$

Often the term $b_0(t)D^0$ is shortened to just $b_0(t)$. Consider a second-order linear differential equation, written as

$$L(x(t)) = (aD^2 + bD^1 + cD^0)x(t) = (a\frac{d^2}{dt^2} + b\frac{d}{dt} + c)x(t) = 0.$$

We guess, that $x(t) = e^{\lambda t}$ is a solution, and note that

$$aD^2(e^{\lambda t}) = a\frac{d^2}{dt^2}(e^{\lambda t}) = a\lambda^2 e^{\lambda t}.$$

So that

$$L(e^{\lambda t}) = (a\lambda^2 + b\lambda + c)e^{\lambda t} = 0.$$

Here the right hand side indicates the usual multiplication of a polynomial by $e^{\lambda t}$. This follow from the context that no D operator appears. Because $e^{\lambda t}$ is never zero, we must have that the so-called **characteristic equation** is equal to zero if the equation is to hold.

Definition 2.

$$p(\lambda) = a\lambda^2 + b\lambda + c = 0$$

with $p(\lambda)$ is called the **characteristic polynomial**. The roots of $p(\lambda)$ are called the **characteristic roots** or **eigenvalues**.

Note: In general, if L has n derivatives, then the **characteristic polynomial** associated with

$$L = D^n + b_{n-1}(t)D^{n-1} + \ldots + b_1(t)D + b_0(t)$$

is

$$p(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_1\lambda + b_0$$

and for any root λ_i of the characteristic equation we have that

$$L(e^{\lambda_i t}) = 0.$$

For any nth degree polynomial there will be exactly n complex roots of this polynomial and hence exactly n exponential solutions of the form $e^{\lambda t}$. Could there be any other roots that are not of this form? It can be shown, and we will mention the method, that this cannot happen in the next lecture.