**Mathematics Homework**

**Seminar on Optimization and Control**

**of Physiological Systems Control**

**Special Research Center for Optimization and Control**
(SFB F003 "Optimierung und Kontrolle")
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1. Show that $f(t) = -sin(2t)$ solves the system
$$\frac{d^2 f(t)}{dt^2} + 4f(t) = 0.$$

2. Given the insulin glucose model (from Math Lecture 1):
$$\begin{aligned}\frac{dg(t)}{dt} &= -m_1 g(t) - m_2 h(t)\\ \frac{dh(t)}{dt} &= +m_3 g(t) - m_4 h(t)\\ &\text{with } m_1 = 1, \quad m_2 = 2, \quad m_3 = 0.5, \quad m_4 = 1\end{aligned}$$
   (a) Find the characteristic equation.
   (b) What stability characteristics will be exhibited by the **zero** steady-state for this system.
   (c) Using the eigenvalues find their associated eigenvectors.
   (d) With the eigenvalues and eigenvectors construct $e^{\mathbf{A}t}$ for this system.
   (e) Using the above results, solve the initial value problem with initial condition
$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

3. What role does linearization play in stability analysis of non linear systems? What are some problems with this method?

4. What are the strengths and weaknesses of the above insulin/glucose model as compared to the model given by Prof. Schneditz in the first Physiology Lecture?

5. Given the following system

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

   (a) Show that the above system controllable.

   (b) Find the **feedback matrix** which transforms the system to one with eigenvalues $\lambda = -1$, and $\lambda = -2$.

6. We consider the delay differential equation

$$\dot{x}\,(t) = -x(t - \frac{\pi}{2}) \qquad (1)$$

$$\varphi(\frac{\pi}{4}) = x(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \qquad (2)$$

$$x(t) = \varphi(t), \ -\frac{\pi}{4} \leq t \leq \frac{\pi}{4} \qquad (3)$$

   Verify that $x(t) = \cos t$ and $x(t) = \sin t$ are both solutions of the system (1) - (3) in $R \times R$

7. **EXTRA CREDIT** We consider now the general case with $r \geq 0$

$$\dot{x}\,(t) = -x(t - r), \ t \geq 0 \qquad (4)$$

$$x(t) = \varphi(t), \ -r \leq t \leq 0 \qquad (5)$$

   and we have this theorem

   **Theorem.** *The delay differential equation system (4) - (5) has one and only one solution in $R \times C([-\frac{\pi}{2}, 0])$ with initial value $\varphi$.*

   Prove that the delay differential system (1)-(3) has also one and only one solution in $R \times C([-\frac{\pi}{2}, 0])$.