On power monoids and their isomorphisms

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Ring Theory Seminar

University of Graz, December 12, 2024

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Outline

1. Power semigroups and power monoids

2. A conjecture by Bienvenu and Geroldinger on power monoids

3. On Power Monoids and their Automorphisms

Preparation

A semigroup is a nonempty set G together with a binary operation on G which is associative: $a(bc) = (ab)c$ for all $a, b, c \in G$.

There are two notations commonly used for the operation of a semigroup:

- the multiplicative notation, where the sgrp operation is called multiplication and denoted by a centered dot $\cdot$ (with or without subscripts or superscripts);
- the additive notation, where the sgrp operation is called addition and denoted by a plus sign $+$ (with or without subscripts or superscripts).
- I'll write all operations multiplicatively unless a statement to the contrary is made.

A monoid is a semigroup G which contains an identity element $e \in G$ such that $ae = ea = a$ for all $a \in G$.

A group is a monoid G such that for every $a \in G$ there exists an inverse element $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$.

Given semigroups G and H, a function $f\colon G \to H$ is a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

If f is bijective, f is called an isomorphism, G and H are said to be isomorphic (written $G \simeq H$). A homomorphism $f\colon G \to G$ is called an isomorphism $f\colon G \to G$ is called an automorphism of G. Under the operation of composition, the automorphisms of S form a group, henceforth called the automorphism group of S and denoted by $\mathrm{Aut}(S)$.

Power semigroups and power monoids

The large power semigroup of a semigroup S is the semigroup $\mathcal{P}(S)$ obtained by endowing the *non-empty* subsets of S with the (provably associative) operation

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\ y \in Y\}.$$

If S is a monoid, then its large power semigroup is itself a monoid with identity $\{1_M\}$ (where 1_M is the identity element of M) and is therefore called the large power monoid of M.

Each of the following is a *unital* submonoid of $\mathcal{P}(M)$:

- $\mathcal{P}_\times(M) := \{X \in \mathcal{P}(M) \colon X \cap M^\times \neq \varnothing\}$, the restricted large PM of M.
- $\mathcal{P}_1(M) := \{X \in \mathcal{P}(M) \colon 1_M \in X\}$, the reduced large PM of M.
- $\mathcal{P}_{\text{fin}}(M) = \{X \in \mathcal{P}(M) \colon |X| < \infty\}$, the finitary PM of M.
- $\mathcal{P}_{\text{fin},\times}(M) := \mathcal{P}_{\text{fin}}(M) \cap \mathcal{P}_\times(M)$, the restricted finitary PM of M.
- $\mathcal{P}_{\text{fin},1}(M) := \mathcal{P}_{\text{fin}}(M) \cap \mathcal{P}_1(M)$, the reduced finitary PM of M.

Altogether, these structures will be referred to as power monoids (PMs).

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The Bienvenu–Geroldinger conjecture

An interesting question: given a class O of monoids, prove/disprove that $\mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},1}(K)$, for some $H, K \in O$, iff $H \simeq K$.

Definition 1.

A numerical monoid is a submonoid of $(\mathbb{N}, +)$ s.t. $\mathbb{N} \setminus S$ is finite.

Bienvenu–Geroldinger conjecture

The reduced finitary power monoid of a numerical monoid S_1 is isomorphic to the reduced finitary power monoid of a numerical monoid S_2 iff $S_1 = S_2$.

The Bienvenu–Geroldinger conjecture was recently settled by Salvatore Tringali and me in a 7-page note (to appear in Proc. AMS).

Main result

The reduced finitary power monoids $\mathcal{P}_{\mathrm{fin},0}(S_1)$ and $\mathcal{P}_{\mathrm{fin},0}(S_2)$ of two Puiseux monoids S_1 and S_2 are isomorphic iff S_1 and S_2 are (a Puiseux monoid is a submonoid of the non-negative rational numbers under addition).

Nathanson's Theorem

For all $a, b \in \mathbb{Z}$ and $\varnothing \neq A \subseteq \mathbb{Z}$, we take $[\![a, b]\!] := \{x \in \mathbb{Z} \colon a \leq x \leq b\}$ to be the discrete interval from a to b, and $\gcd A$ to be the greatest common divisor of A. That is, $\gcd A$ is the largest non-negative divisor of each $a \in A$, where the term "largest" refers to the divisibility preorder on the multiplicative monoid of the integers. In particular, note that $\gcd\{0\} = 0$.

Nathanson's Theorem (or Fundamental Theorem of Additive NT)

Given $A \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ with $\gcd A = 1$, there exist $b, c \in \mathbb{N}$, $B \subseteq [\![0, b-2]\!]$, and $C \subseteq [\![0, c-2]\!]$ s.t., for all large $k \in \mathbb{N}$,

$$kA = B \cup [\![b, ka - c]\!] \cup (ka - C),$$

where $a := \max A$ and $kA := A + \cdots + A$ (k times).

Example 2.

$A = \{0, 2, 5\}$
$2A = \{0, 2, 4, 5, 7, 10\}$
$3A = \{0, 2, 4, 5, 6, 7, 9, 10, 12, 15\}$
$4A = \{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 20\}$
$\ldots$
$kA = \{0, 2\} \cup [\![4, 5k - 8]\!] \cup (5k - \{0, 3, 5, 6\})$

Lemma 3.

If $A \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{Q}_{\geq 0})$, then $(k+1)A = kA + \{0, \max A\}$ for all large $k \in \mathbb{N}$.

Proof.

- Since A is a non-empty finite subset of $\mathbb{Q}_{\geq 0}$ containing 0, there exist $d \in \mathbb{N}^+$ and $A' \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ such that $A = A'/d$. It follows that $\max A = \max A'/d$ and $kA = kA'/d$ for all $k \in \mathbb{N}$, which makes it possible to assume $d = 1$ and hence $A \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$.
- By Nathanson's Theorem, there exist non-negative integers b, c, and k_0 and sets $B \subseteq [\![0, b-2]\!]$ and $C \subseteq [\![0, c-2]\!]$ such that $kA = B \cup [\![b, ka - c]\!] \cup (ka - C)$, for each integer $k \geq k_0$.
- Fix an integer $h \geq \max\{k_0, 1 + (b+c)/a\}$. It is obvious that $hA + \{0, a\} \subseteq hA + A = (h+1)A$.
- For the inverse, we have
  $hA = B \cup [\![b, ha - c]\!] \cup (ha - C)$ and
  $(h+1)A = B \cup [\![b, (h+1)a - c]\!] \cup ((h+1)a - C)$.

  Let $x \in (h+1)A$. Either $x \in B \cup [\![b, ha - c - 1]\!]$, and then $x \in hA$; or $x - a \in [\![(h-1)a - c, ha - c]\!] \cup (ha - C) \subseteq hA$, and then $x \in hA + a$, where we are especially using that $h \geq 1 + (b+c)/a$ and hence $(h-1)a - c \geq b$. In both cases, $x \in hA + \{0, a\}$, which finishes the proof as x is arbitrary in $(h+1)A$. ∎

The proof

Lemma 4.

An isomorphism $\phi\colon \mathcal{P}_{\text{fin},0}(S_1) \to \mathcal{P}_{\text{fin},0}(S_2)$, where S_1 and S_2 are Puiseux monoids, sends 2-element sets to 2-element sets.

Proof.

- Fix a non-zero $a \in S_1$. We need to show that $B := \phi(\{0,a\}) = \{0,b\}$ for some (non-zero) $b \in S_2$.
- Set $b := \max B \in S_2$ and note that b is non-zero, there then exists an integer $k \geq 0$ such that $(k+1)B = kB + \{0,b\}$.
- Put $A := \phi^{-1}(\{0,b\})$, where ϕ^{-1} is the inverse of ϕ. Since ϕ^{-1} is an isomorphism $\mathcal{P}_{\text{fin},0}(S_2) \to \mathcal{P}_{\text{fin},0}(S_1)$ with $\phi^{-1}(B) = \{0,a\}$, we get from the above that

$$(k+1)\{0,a\} = (k+1)\phi^{-1}(B) = k\phi^{-1}(B) + \phi^{-1}(\{0,b\}) = k\{0,a\} + A.$$

- It follows that $\{0\} \subsetneq A \subseteq (k+1)\{0,a\}$ and $\max A = (k+1)a - ka = a$. So, noticing that a is the least non-zero element of $(k+1)\{0,a\}$, we find $A = \{0,a\}$ and hence $B = \phi(\{0,a\}) = \phi(A) = \{0,b\}$. ■

Lemma 5.

Let $\phi\colon \mathcal{P}_{\text{fin},0}(S_1) \to \mathcal{P}_{\text{fin},0}(S_2)$ be an isomorphism, where S_1 and S_2 are Puiseux monoids, and pick $a_1, a_2 \in S_1$. The following hold:

1. There exists $b_i \in S_2$ such that $\phi(\{0, a_i\}) = \{0, b_i\}$ $(i = 1, 2)$.
2. $\phi(\{0, a_1 + a_2\}) = \{0, b_1 + b_2\}$.

Proof.

- Define $A := \{0, a_1\} + \{0, a_2\}$, $B := \phi(A)$, and $a_0 := a_1 + a_2 = \max A \in S_1$. Then for each $i \in [\![0, 2]\!]$, there is an element $b_i \in S_2$ such that $\phi(\{0, a_i\}) = \{0, b_i\}$.
- We know that $(k+1)A = kA + \{0, a_0\}$ for some $k \in \mathbb{N}$. Since $\phi(X + Y) = \phi(X) + \phi(Y)$ for all $X, Y \in \mathcal{P}_{\text{fin},0}(S_1)$, it is thus found that

$$B = \phi(A) = \phi(\{0, a_1\}) + \phi(\{0, a_2\}) = \{0, b_1\} + \{0, b_2\}$$

and

$$(k+1)B = (k+1)\phi(A) = \phi((k+1)A) = k\phi(A) + \phi(\{0, a_0\}) = kB + \{0, b_0\}.$$

Consequently, $b_0 = (k+1)\max B - k \max B = \max B = b_1 + b_2$ (as wished). ■

The proof

Theorem 6.

The reduced finitary power monoids $\mathcal{P}_{\mathrm{fin},0}(S_1)$ and $\mathcal{P}_{\mathrm{fin},0}(S_2)$ of two Puiseux monoids S_1 and S_2 are isomorphic iff S_1 and S_2 are.

Proof.

- The "if" part: let $f\colon S_1 \to S_2$ be a monoid isomorphism, and let F be the function

  $$\mathcal{P}_{\mathrm{fin},0}(S_1) \to \mathcal{P}_{\mathrm{fin},0}(S_2)\colon X \mapsto f[X],$$

  where $f[X] := \{f(x)\colon x \in X\} \subseteq S_2$ is the (direct) image of X under f.
- The "only if" part: let ϕ be an isomorphism $\mathcal{P}_{\mathrm{fin},0}(S_1) \to \mathcal{P}_{\mathrm{fin},0}(S_2)$. ϕ maps a 2-element set $\{0, a\} \subseteq S_1$ to a 2-element set $\{0, b\} \subseteq S_2$.
- Conversely, any 2-element set $\{0, b\} \subseteq S_2$ is the image under ϕ of a 2-element set $\{0, a\} \subseteq S_1$, because the inverse ϕ^{-1} of ϕ is itself an isomorphism, with the result that, for each non-zero $b \in S_2$, there is a non-zero $a \in S_1$ with $\phi^{-1}(\{0, b\}) = \{0, a\}$.
- It follows that the function $\Phi\colon S_1 \to S_2\colon a \mapsto \max \phi(\{0, a\})$ is bijective; and on the other hand, we get that Φ is a homomorphism (from S_1 to S_2). ■

The proof

Corollary 7.

The reduced finitary power monoids of two numerical monoids S_1 and S_2 are isomorphic iff $S_1 = S_2$.

Proof.

All that remains is to prove that two numerical monoids are isomorphic iff they are equal, see J. C. Higgins, *Representing N-semigroups*, Bull. Austral. Math. Soc. **1** (1969), 115–125, Theorem 3. ■

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Main result

It's an interesting problem to study the automorphism group of any mathematical object. In particular, it's interesting to do so for the group of semigroup automorphisms of $\mathcal{P}_{\mathrm{fin},1}(H)$, where H is a monoid. Here we will consider the case of $(\mathbb{N}, +)$.

Main result

The only automorphisms of the reduced power monoid $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ of $(\mathbb{N}, +)$ are the identity $X \mapsto X$ and the reversion map $X \mapsto \max X - X$.

The proof

Theorem 8.

The following are equivalent for a homomorphism $f\colon \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}) \to \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$:

1. f is injective and $f(\{0,1\}) = \{0,1\}$.
2. f is surjective.
3. f is an automorphism.

Corollary 9.

For an automorphism f of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, the following hold:

1. $\max X = \max f(X)$ for every $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$.
2. $\{0,k\}$ and $[\![0,k]\!]$ are fixed points of f for all $k \in \mathbb{N}$.
3. Either $f(\{0,2,3\}) = \{0,1,3\}$ or $f(\{0,2,3\}) = \{0,2,3\}$.

The proof

One consequence of Corollary 9 is that any automorphism f of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$ gives rise to a well-defined function $f^*\colon \mathcal{P}_{\text{fin},0}(\mathbb{N}) \to \mathcal{P}_{\text{fin},0}(\mathbb{N})$, henceforth referred to as the reversal of f, by taking $f^*(X) := \max X - f(X)$ for all $X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$. The next lemma shows that something more is true.

Lemma 10.

The reversal of an automorphism of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$ is itself an automorphism.

Proof.

- Let f be an automorphism of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$, and let $X, Y \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$.
- $f^*(X+Y) = \max(X+Y) - f(X+Y) = (\max X - f(X)) + (\max Y - f(Y)) = f^*(X) + f^*(Y)$. We are left to see that f^* is a bijection:
- Injectivity: If $f^*(X) = f^*(Y)$ for some $X, Y \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$, then $\max X - f(X) = \max Y - f(Y)$. Since $\max(\max Z - f(Z)) = \max Z$ for all $Z \in P_{\text{fin},0}(\mathbb{N})$. It follows that $\max X = \max Y$ and hence $f(X) = f(Y)$. By the injectivity of f, we can get $X = Y$.
- Surjectivity: Let $Y \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$. Since f is surjective, there exists $X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$ such that $f(X) = \max Y - Y$. By Corollary 9, we have $\max X = \max f(X) = \max Y$ and hence $Y = \max X - f(X) = f^*(X)$. ■

The proof

Given $X \subseteq \mathbb{Z}$, we denote by $\Delta(X)$ the gap set of X, i.e., the set of all integers $d \geq 1$ such that $\{x, x+d\} = X \cap [\![x, x+d]\!]$ for some $x \in \mathbb{Z}$.

Lemma 11.

If f is an automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, then $\max \mathrm{gap}(X) = \max \mathrm{gap}(f(X))$ for all $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$.

Proof.

- Let $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ and put $d := \max \mathrm{gap}(f(X))$. Since the functional inverse of f is itself an automorphism of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, it suffices to prove that $\max \mathrm{gap}(X) \leq d$.
- Set $X' := X + [\![0, d-1]\!]$ and suppose to the contrary that $d < \max \mathrm{gap}(X)$. Since $f(\{0\}) = \{0\}$, the gap set of X is then a non-empty finite subset of $\mathbb{N}^+$ and hence d is a *positive* integer.
- $f(X) + [\![0, d-1]\!] = [\![0, d-1+\max f(X)]\!] = [\![0, d-1+\max X]\!]$.
- $f(X') = f(X) + f([\![0, d-1]\!]) = f(X) + [\![0, d-1]\!] = f([\![0, d-1+\max X]\!])$.
- So, we conclude that $X' = [\![0, d-1+\max X]\!]$, it's a contradiction. ■

The proof

Proposition 12.

For all $a, n \in \mathbb{N}$ with $n \geq a+1$, it holds that

$$\sum_{i=0}^{n-1} \{0, a+i, a+i+1\} = \{0\} \cup \left[\!\left[a, na + \tfrac{1}{2}n(n+1) \right]\!\right]. \tag{1}$$

The proof

Lemma 13.

Assume $\{0,2,3\}$ is a fixed point of an automorphism f of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$. The following hold:

1. If $1 \in X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$, then $1 \in f(X)$.
2. $f(\{0,a,a+1\}) = \{0,a,a+1\}$ for every $a \in \mathbb{N}$.
3. $\{0\} \cup \llbracket a, na + \frac{1}{2}n(n+1) \rrbracket$ is a fixed point of f for all $a, n \in \mathbb{N}$ with $n \geq a+1$.

Proof of 1.

- Let $X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$ and $k \in \mathbb{N}$. Then we have $k\{0,2,3\} = \{0\} \cup \llbracket 2, 3k \rrbracket$.
- If $1 \in X$ and $k \geq (2 + \max X)/3$, then $\llbracket 0, 3k + \max X \rrbracket = (\{0,1\} \cup \{\max X\}) + (\{0\} \cup \llbracket 2, 3k \rrbracket) \subseteq X + k\{0,2,3\} \subseteq \llbracket 0, 3k + \max X \rrbracket$ that is, $X + k\{0,2,3\}$ is the interval $\llbracket 0, 3k + \max X \rrbracket$ and hence it's a fixed point of f.
- We conclude that, if $1 \in X$ and k is a sufficiently large integer, then

$$1 \in \llbracket 0, 3k + \max X \rrbracket = f(X + k\{0,2,3\}) = f(X) + k\{0,2,3\} = f(X) + (\{0\} \cup \llbracket 2, 3k \rrbracket)$$

■

The proof

Proof of 2 and 3.

- Let $a \in \mathbb{N}$, and set $X := \{0, a, a+1\}$ and $Y := f(X)$. We need to prove $Y = X$. If $a = 0$ or $a = 1$, then X is an interval and we are done. So, assume $a \geq 2$.
- $\max Y = \max X = a + 1$ and $\delta := \max \operatorname{gap}(Y) = \max \operatorname{gap}(X) = a \geq 2$. It follows that Y does not contain any integer y in the interval $[\![2, a-1]\!]$, or else we would find that $\delta \leq \min(y, a+1-y) \leq a - 1 < \delta$ (a contradiction).
- $\{0, a+1\} \subsetneq Y \subseteq \{0, a, a+1\} = X$, which shows that $Y = X$ and completes the proof of 2.
- Given $a, n \in \mathbb{N}$ with $n \geq a+1$, we have from Proposition 12 that, $\{0\} \cup \left[\!\left[a, na + \frac{1}{2}n(n+1)\right]\!\right]$ can be written as the sum of the sets $\{0, a+i, a+i+1\}$ as i ranges over the interval $[\![0, n-1]\!]$, this is enough to prove the claim. ■

The proof

Given a set $S \subseteq \mathbb{Z}$, we denote by $\mathrm{b.dim}(S)$ the smallest integer $k \geq 0$ for which there exist k (discrete) intervals whose union is S, with the understanding that if no such k exists then $\mathrm{b.dim}(S) := \infty$. We call $\mathrm{b.dim}(S)$ the boxing dimension of S.

Theorem 14.

The only automorphisms of the reduced power monoid $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ of $(\mathbb{N}, +)$ are the identity $X \mapsto X$ and the reversion map $X \mapsto \max X - X$.

Proof.

- Let Γ be the set of automorphisms of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ that fix $\{0, 2, 3\}$, and define $\Gamma' := \{f^* : f \in \Gamma\}$. We infer that $\mathrm{Aut}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})) = \Gamma \cup \Gamma'$. It is therefore enough to show that the only automorphism in Γ is the identity $X \mapsto X$.
- For, let $f \in \Gamma$ and $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, and put $Y := f(X)$, $r := \mathrm{b.dim}(X) - 1$, $s := \mathrm{b.dim}(Y) - 1$, and $t := \min(r, s)$. We get that $\mu := \max(X) = \max(Y)$.
- There exist increasing sequences $x_0, x_1, \ldots, x_{2r+1}$ and $y_0, y_1, \ldots, y_{2s+1}$ of integers such that
  1. $x_{2i-1} + 2 \leq x_{2i}$ for each $i \in [\![1, r]\!]$ and $y_{2j-1} + 2 \leq y_{2j}$ for each $j \in [\![1, s]\!]$;
  2. $X = [\![x_0, x_1]\!] \cup \cdots \cup [\![x_{2r}, x_{2r+1}]\!]$ and $Y = [\![y_0, y_1]\!] \cup \cdots \cup [\![y_{2s}, y_{2s+1}]\!]$.

  In particular, $x_0 = y_0 = 0$ and $x_{2r+1} = y_{2s+1} = \mu$. We will prove by induction on r that $X = Y$. ■

The proof

Proof.

- If $r = 0$, then X is an interval and we have $X = f(X) = Y$. So, let $r \geq 1$, assume for the sake of induction that $f(S) = S$ for all $S \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$ with $\text{b.dim}(S) \leq r$, and suppose by way of contradiction that $X \neq Y$. Accordingly, there is a smallest index $v \in [\![1, 2t+1]\!]$ such that $x_v \neq y_v$; otherwise, since $x_0 \leq x_1 \leq \cdots \leq x_{2r+1}$ and $y_0 \leq y_1 \leq \cdots \leq y_{2s+1}$, we would get that $X = Y$, which is absurd. We distinguish two cases, depending on whether v is even or odd.
- To start with, there is no loss of generality in assuming $x_v < y_v$; otherwise, we could replace f with its functional inverse f^{-1}. And it is clear that $r \leq s$, or else Y is fixed by f.
- Case 1: $v = 2u$ for some $u \in [\![1, r]\!]$. Put $d := x_{2u} - x_{2u-1} - 1$ and $I := [\![0, r]\!] \setminus \{u-1, u\}$, and set $X_1 := [\![x_{2(u-1)}, x_{2u+1} + d]\!]$ and $X_2 := \bigcup_{i \in I} [\![x_{2i}, x_{2i+1} + d]\!]$.
- $X + [\![0, d]\!] = \bigcup_{i=0}^{r} ([\![x_{2i}, x_{2i+1}]\!] + [\![0, d]\!]) = \bigcup_{i=0}^{r} [\![x_{2i}, x_{2i+1} + d]\!] = X_1 \cup X_2$.
- In a similar way, $Y + [\![0, d]\!] = \bigcup_{j=0}^{s} [\![y_{2j}, y_{2j+1} + d]\!]$.
- It is obvious that $\text{b.dim}(X \cup Y) \leq \text{b.dim}(X) + \text{b.dim}(Y)$ for all $X, Y \subseteq \mathbb{Z}$. Then $\text{b.dim}(X + [\![0, d]\!]) \leq \text{b.dim}(X_1) + \text{b.dim}(X_2) \leq 1 + |I| = r < \text{b.dim}(X)$.
- $X + [\![0, d]\!] = f(X + [\![0, d]\!]) = f(X) + f([\![0, d]\!]) = Y + [\![0, d]\!]$. This is however impossible, because x_{2u} is an element of $X + [\![0, d]\!]$ but not of $Y + [\![0, d]\!]$. ∎

Ongoing research

Kerou Wen and Salvatore Tringali have recently announced a closely related result: that the automorphism group of $\mathcal{P}_{\text{fin}}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2 \times \text{Dih}_\infty$ (the direct product of the cyclic group of order 2 by the infinite dihedral group).

Conjecture 15.

The automorphism group of the reduced power monoid of a numerical monoid *properly* contained in $\mathbb{N}$ is trivial (that is, the only automorphism is the identity).

Question.

For which groups G does there exist a monoid H such that the automorphism group of the reduced power monoid $\mathcal{P}_{\text{fin},1}(H)$ of H is isomorphic to G?