Power monoids and a conjecture by Bienvenu and Geroldinger

Salvatore Tringali

School of Mathematical Sciences, Hebei Normal University

based on joint work with Weihao Yan[1]

Algebra & Number Theory seminar

University of Graz, IMSC $\sim$ Graz (AT), Mar 30, 2023

[1] To date, Weihao is an undergrad student in mathematics at Hebei Normal University.

Outline

What is. . . a power monoid?

Throughout, H is a multiplicative[ly written] mon (short for "monoid") and we denote by $H^\times$ its group of units; H need not be commutative, cancellative, etc.

$\mathcal{P}(H)$: the mon obtained by endowing the family of all *non-empty* (finite or infinite) subsets of H with the operation of setwise multiplication

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\ y \in Y\}.$$

Each of the following is a submon of $\mathcal{P}(H)$:

- $\mathcal{P}_\times(H) := \{X \in \mathcal{P}(H) \colon X \cap H^\times \neq \varnothing\}$.
- $\mathcal{P}_1(H) := \{X \in \mathcal{P}(H) \colon 1_H \in X\}$.
- $\mathcal{P}_{\rm fin}(H) := \{X \in \mathcal{P}(H) \colon |X| < \infty\}$.
- $\mathcal{P}_{{\rm fin},\times}(H) := \mathcal{P}_{\rm fin}(H) \cap \mathcal{P}_\times(H)$ and $\mathcal{P}_{{\rm fin},1}(H) := \mathcal{P}_{\rm fin}(H) \cap \mathcal{P}_1(H)$.

Altogether, these structures will be generically called power mons (PMs).

In particular, $\mathcal{P}(H)$ is the large PM, $\mathcal{P}_{\rm fin}(H)$ is the small PM, $\mathcal{P}_{{\rm fin},\times}(H)$ is the restricted small PM, and $\mathcal{P}_{{\rm fin},1}(H)$ is the reduced small PM of H.

Literature and popularization

PMs were introduced by Yushuang Fan and T. in 2018. To date, there are still very few publications devoted to their study (*good news for the young!*):

- Fan & T., J. Algebra **512** (2018), 252–294.
- Antoniou & T., Pacific J. Math. **312** (2021), No. 2, 279–308.
- Sect. 4.2 in T., J. Algebra **602** (July 2022), 352–380.
- Bienvenu & Geroldinger, Israel J. Math., to appear (arXiv:2205.00982).
- Example 4.5(3) and Remark 5.5 in Cossu & T., J. Algebra, to appear (arXiv:2208.05869).

PMs are also the subject of a CrowdMath project recently launched by F. Gotti on the Art of Problem Solving (AoPS) website:

`https://artofproblemsolving.com/polymath/mitprimes2023`

A major problem (from Sect. 5 of [Fan & T., 2018]) is the following:

Conjecture. If H is linearly orderable[2], then every non-empty *finite* subset L of $\mathbb{N}_{\geq 2}$ is the length set of a set $X \in \mathcal{P}_{\mathrm{fin},1}(H)$, i.e., L is the set of all and only the integers $k \geq 0$ such that X is a product of k atoms[3] of $\mathcal{P}_{\mathrm{fin},1}(H)$.

As noted in [Fan & T., 2018], the conjecture boils down to $H = (\mathbb{N}, +)$.

(2) There is a total order $\preceq$ on H s.t. if $x \prec y$ then $uxv \prec uyv$ for all $u, v \in H$.

(3) In a multiplicative mon, an atom is a non-unit not factoring as a product of two non-units.

Why studying PMs

1) Owing to their "high non-cancellativity", PMs are a leading example in the (ongoing) development of a *unifying theory of factorization*, where the role of mons (and atoms) in the *classical* theory is taken up by premons (and irreds):

- T., J. Algebra **602** (July 2022), 352–380.
- Cossu & T., Israel J. Math., to appear (arXiv:2108.12379).
- Cossu & T., J. Algebra, to appear (arXiv:2208.05869).
- T., Math. Proc. Cambridge Philos. Soc., to appear (arXiv:2209.05238).
- Cossu & T., preprint (under review, arXiv:2301.09961).

2) PMs are a natural algebraic framework for *famous* problems in additive NT:

- **Sarkozy's conjecture**[(4)]**.** For all but finitely many primes p, the set $\mathcal{Q}_p \subseteq \mathbb{F}_p$ of quadratic residues mod p is an atom in the small/large PM of the additive group of $\mathbb{F}_p$.
- **Inverse Goldbach conjecture**[(5)]**.** Every set of integers that differ from the set of (positive rational) primes by finitely many elements is an atom in the large PM of $(\mathbb{Z}, +)$.

3) The mon of non-empty (2-sided) ideals of H is a submon of $\mathcal{P}(H)$, which is at least useful to demystify certain ideas and put them in the right perspective.

(4) Conjecture 1.6 in A. Sárközy, Acta Arith. **155** (2012), No. 1, 41–51.

(5) See C. Elsholtz, Mathematika **48** (2001), Nos. 1–2, 151–158.

A zoo of wild beasts

$\mathcal{P}(H)$, $\mathcal{P}_\times(H)$, and $\mathcal{P}_0(H)$ are rather complicated objects — their "finitary analogues" are much tamer, although $\mathcal{P}_{\rm fin}(H)$ can still be a real headache.

In the below diagram, a "hooked arrow" $P \hookrightarrow Q$ means the inclusion map from P to Q and a "tailed arrow" $P \rightarrowtail Q$ means the embedding $P \to Q \colon x \mapsto \{x\}$.

$$
\begin{array}{ccccc}
\{1_H\} & \hookrightarrow & H^\times & \hookrightarrow & H \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{\rm fin,1}(H) & \hookrightarrow & \mathcal{P}_{\rm fin,\times}(H) & \hookrightarrow & \mathcal{P}_{\rm fin}(H) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_1(H) & \hookrightarrow & \mathcal{P}_\times(H) & \hookrightarrow & \mathcal{P}(H)
\end{array}
$$

Fact 1. TFAE:

- $\mathcal{P}_{\rm fin,\times}(H)$ is a divisor-closed submon of $\mathcal{P}_{\rm fin}(H)$.
- $\mathcal{P}_\times(H)$ is a divisor-closed submon of $\mathcal{P}(H)$.
- H is Dedekind-finite.

Fact 2. If H is cancellative, then $\mathcal{P}_{\rm fin}(H)$ is a divisor-closed submon of $\mathcal{P}(H)$.

Fact 3. If H is Dedekind-finite, then $\mathcal{P}_{\rm fin,1}(H)$ and $\mathcal{P}_{\rm fin,\times}(H)$ have the same length sets (relative to factorizations into irreds) and so do $\mathcal{P}_1(H) \hookrightarrow \mathcal{P}_\times(H)$.

Pivots

The FACTS mentioned on the previous slide suggest that, at least for a *Dedekind-finite* H, there is much about $\mathcal{P}(H)$ and other PMs that we can understand from the investigation of $\mathcal{P}_{\text{fin},1}(H)$. In addition:

FACT 4 (Proposition 3.2(iii) in [Antoniou & T., 2019]): $\mathcal{P}_{\text{fin},1}(K)$ is a divisor-closed submon of $\mathcal{P}_{\text{fin},1}(H)$ for *every* submon K of H.

$\implies$ It is a good idea to investigate various properties of $\mathcal{P}_{\text{fin},1}(H)$ when H is a monogenic monoid (i.e., is a generated by one of its elements).

We are thus naturally led to consider

- the reduced PM of $(\mathbb{N}, +)$, herein denoted by $\mathcal{P}_{\text{fin},0}(\mathbb{N})$ and written additively;
- the restricted PM of the group $(\mathbb{Z}/n\mathbb{Z}, +)$, herein denoted by $\mathcal{P}_{\text{fin},0}(\mathbb{Z}/n\mathbb{Z})$.

[When H is *cancellative*, there are no other monogenic submons (up to iso).]

So far, most of the work on PMs has been limited to their arithmetic[6].

P. Bienvenu & A. Geroldinger have recently addressed ideal-theoretic and analytic properties of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$ and closely related structures.

[6] In particular, the arithmetic of $\mathcal{P}_{\text{fin},0}(\mathbb{N})$ is the object of Sect. 4 in [Fan & T., 2018], and the arithmetic of $\mathcal{P}_{\text{fin},0}(\mathbb{Z}/n\mathbb{Z})$ for an *odd* modulus n the object of Sect. 5 in [Antoniou & T., 2019].

Outline

1. Power monoids

2. The Bienvenu–Geroldinger conjecture

3. References

In more detail

Let S be a numerical mon, i.e., a submon of $(\mathbb{N}, +)$ s.t. $\mathbb{N} \setminus S$ is finite.

Among other things, Bienvenu & Geroldinger have

- obtained quantitative results on the "density" of the atoms of the reduced PM of S, herein denoted by $\mathcal{P}_{\mathrm{fin},0}(S)$ and written additively;
- started a foray into the ideal theory of $\mathcal{P}_{\mathrm{fin},0}(S)$, with emphasis on prime ideals.

Moreover, they have formulated (and proved special cases of) the following:

Bienvenu–Geroldinger (BG) conjecture

The reduced PM of a numerical mon S_1 is isomorphic to the reduced PM of a numerical mon S_2 iff $S_1 = S_2$.

It is worth noting that

i) a mon hom $f\colon H \to K$ yields a well-defined (mon) hom $F\colon \mathcal{P}_{\mathrm{fin},1}(H) \to \mathcal{P}_{\mathrm{fin},1}(K)\colon X \mapsto f(X)$, and if f is iso then so also is F;

ii) the converse of i) need not be true — if H is an idempotent monoid with two elements, then $H \not\simeq (\mathbb{Z}/2\mathbb{Z}, +)$ but $H \simeq \mathcal{P}_{\mathrm{fin},1}(H) \simeq \mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}/2\mathbb{Z})$.

iii) the BG conjecture is ultimately asking to show that i) can be reversed when H and K numerical mons, as it is folklore[7] that two numerical mons are isomorphic iff they are equal.

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[7] See Theorem 3 in J. C. Higgins, Bull. Austral. Math. Soc. **1** (1969), 115–125.

Proof outline

The proof of the BG conjecture is elementary and, *in hindsight*, quite simple — the most "advanced" result we use is a classical result[8] commonly known as

Fundamental Theorem of Additive Combinatorics

Given $A \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ with $\gcd A = 1$, there exist $b, c \in \mathbb{N}$, $B \subseteq [\![0, b-2]\!]$, and $C \subseteq [\![0, c-2]\!]$ s.t. $kA = B \cup [\![b, ka-c]\!] \cup (ka - C)$ for all large $k \in \mathbb{N}$, where $a := \max A$ and $kA := A + \cdots + A$ (k times).

One can break down the proof to the following steps:

1) Prove by the Fundamental Theorem of Additive Combinatorics that, given $A \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, we have $(k+1)A = kA + B$ for all large $k \in \mathbb{N}$ and every $B \subseteq A$ with $\{0, \max A\}$.

2) Use 1) to show that an injective endo of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ sends 2-element sets to 2-element sets.

3) Use 2) to prove that, if ϕ is an *injective* homo from the reduced PM of a numerical mon S to $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$ and $a_1, \ldots, a_n$ are elements in S, then

(i) there exist $b_1, \ldots, b_n \in \mathbb{N}$ s.t. $\phi(\{0, a_i\}) = \{0, b_i\}$ for each $i \in [\![1, n]\!]$.

(ii) $\phi(\{0, a_1 + \cdots + a_n\}) = \{0, b_1 + \cdots + b_n\}$.

4) Use 3) to conclude that, if S_1 and S_2 are numerical mons and ϕ is an iso $\mathcal{P}_{\mathrm{fin},0}(S_1) \to \mathcal{P}_{\mathrm{fin},0}(S_2)$, then the fnc $\Phi\colon S_1 \to S_2 : a \mapsto \max \phi(\{0, a\})$ is itself a (mon) iso.

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[8] See M. B. Nathanson, Amer. Math. Monthly **79** (1972), No. 9, 1010–1012.

Outline

Bibliography

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