January 11, 2024

Matrix-rank approaches to factorization problems in rings of integer-valued polynomials

FWF Der Wissenschaftsfonds.

UNIVERSITÄT KLAGENFURT

Modeling - Analysis - Optimization DOC.FUNDS DOCTORAL SCHOOL

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$$x^3 - x = x(x-1)(x+1)$$

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Factorizations and $\mathrm{Int}(\mathbb{Z})$

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

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$$2 \cdot 3 \cdot 7 \cdot \binom{x}{42} = \binom{x}{41} \cdot (x-41)$$

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How to recognize **?**

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$\binom{x}{n}$ is irreducible in $\mathrm{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.

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Theorem (R., Windisch; 2021)

$\binom{x}{n}$ is **absolutely** irreducible in $\mathrm{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.

The binomial polynomials

$$\binom{x}{n}^{k} = \left(\frac{x(x-1)\cdots(x-n+1)}{n!}\right)^{k} = (\pm 1)\cdot\left(\pm\binom{x}{n}\right)^{k}$$

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$$\binom{x}{3}^k = f \cdot g \quad \Longrightarrow \quad f = \pm \left(\frac{x}{3}\right)^{k_0} \left(\frac{x-1}{2}\right)^{k_1} \left(\frac{x-2}{1}\right)^{k_2}$$

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for certain s

The valuation matrix

$$A_n = \left(\mathsf{v}_p(n+r-i) - \mathsf{v}_p(n-i)\right)_{\substack{p\in\mathcal{P}_n, r\in\mathcal{R}_{n,p} \\ 0\le i\le n-1}}$$

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Theorem (R., Windisch; 2021)

- $\prod_{i=0}^{n-1}\left(\frac{x-i}{n-i}\right)^{k_i}$ divides $\binom{x}{n}^m \Longrightarrow (k_0, k_1, \ldots, k_{n-1})^t \in \ker(A_n)$

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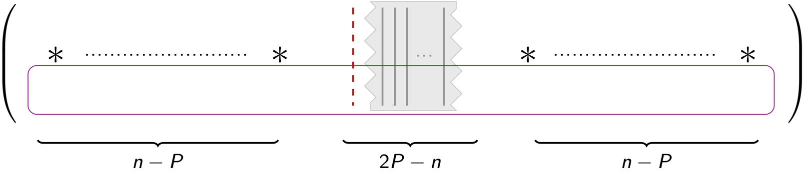
- $\operatorname{rank}(A_n) = n-1 \implies \binom{x}{n}$ absolutely irreducible

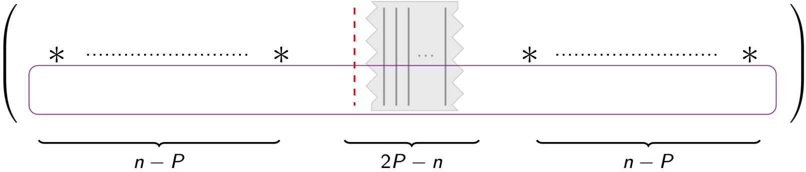
The valuation matrix

$$\left(\underbrace{* \;\cdots\cdots\cdots\; *}_{n-P} \quad \underbrace{\;\cdots\;}_{2P-n} \quad \underbrace{* \;\cdots\cdots\cdots\; *}_{n-P} \right)$$

The valuation matrix

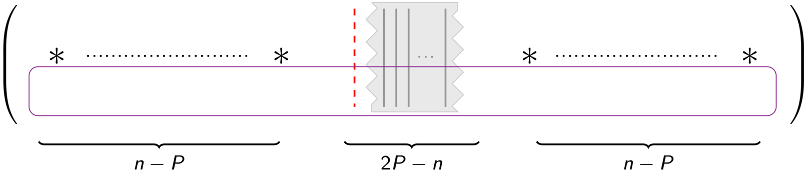


The valuation matrix



$$\begin{pmatrix} 0 & \boxed{\begin{matrix}-1\\ \vdots \\ -1\end{matrix}} & \boxed{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}} & 0\end{pmatrix}$$

The valuation matrix



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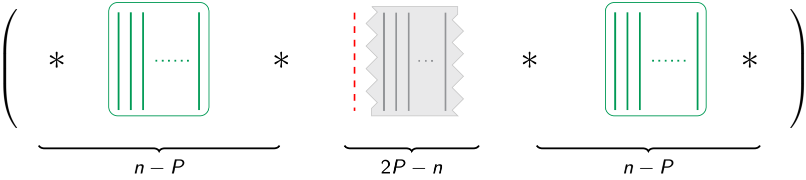
- “inner” rank is $2P-n-1$
- “inner” and “outer” columns spaces have trivial intersection

The valuation matrix



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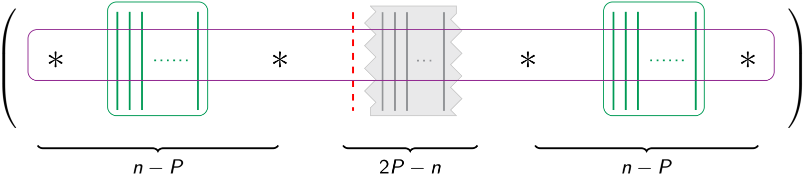

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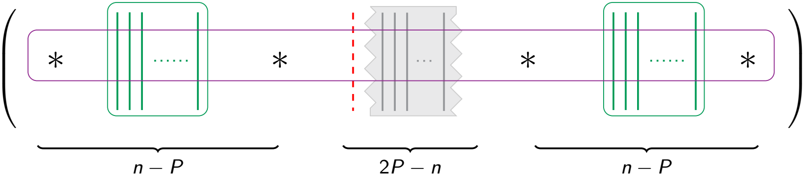
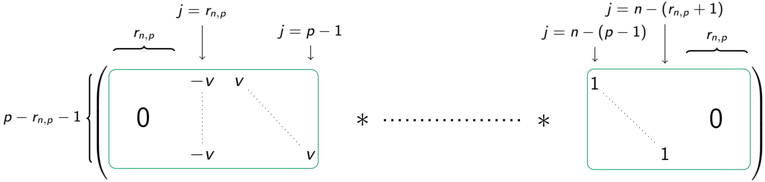
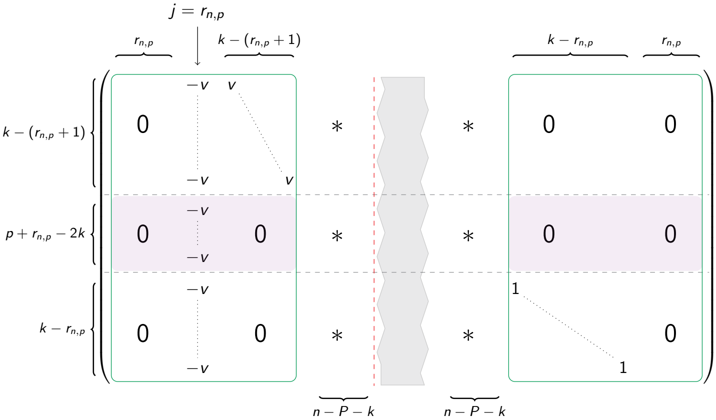




The "outer" rank



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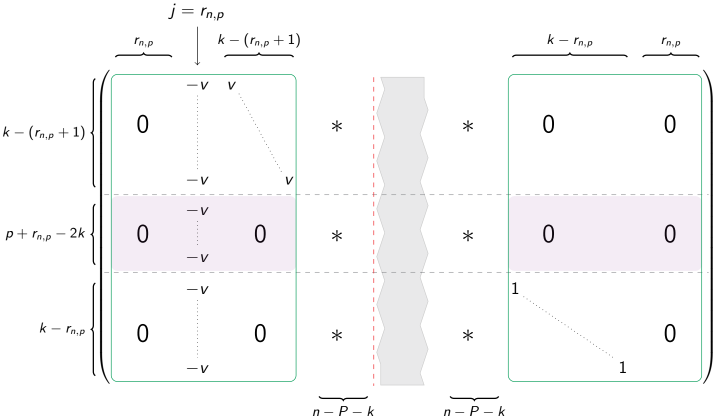

Theorem (R., Windisch, 2021)

Let $n > 10$, P the maximal prime with $P \leq n$, and $2 \leq k \leq n - P$.

Then there exists a prime $p > 2k$ with $p \mid n(n-1)\cdots(n-k+1)$.

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All int.-val. polynomials over DVRs

$(R, pR) \ldots$ DVR
$\mathrm{Int}(R) = \{f \in q(R)[x] \mid f(R) \subseteq R\}$

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Lemma

For all $a \in R$ with $\mathrm{v}(f(a)) = n$ exists a $w \in W$ with

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$$k + \ell \geq (n+1)\left(\left\lceil \max\left\{\frac{v_g}{m_g} : v_g > 0\right\}\right\rceil + \left\lceil \max\left\{\frac{-v_g}{m_g} : v_g < 0\right\}\right\rceil\right)$$

$$\min\{\|\boldsymbol{v}\|_\infty : 0 \neq \boldsymbol{v} \in \text{fd-ker}(f) \cap \mathbb{Z}^{\mathcal{P}}\} \leq n^{\mathrm{rank}(A)-u}$$

Theorem (Hiebler, Nakato, R., 2023)

The following are equivalent for irreducible $F \in \mathrm{Int}(R)$:

- F is absolutely irreducible
- F^S factors uniquely for $S = 2(n+1)n^{\mathrm{rank}(A)-u}$
- F^S factors uniquely for $S = 2(n+1)n^{q^{\lceil \frac{n}{2} \rceil}}$

? Next:

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Next:

- Local vs. global (one vs. many primes)
- What are the limits of the non-uniqueness of these factorizations?
- How many different factorizations? Of which lengths?
- Construct factorizations of powers of irreducibles

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