



January 11, 2024

Matrix-rank approaches to factorization problems in rings of integer-valued polynomials

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Factorizations

$$42 = 2 \cdot 3 \cdot 7$$

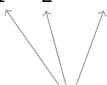
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Factorizations and $\text{Int}(\mathbb{Z})$

$$\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

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$$2 \cdot 3 \cdot 7 \cdot \binom{x}{42} = \binom{x}{41} \cdot (x-41)$$

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How to recognize ?

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$\binom{x}{n}$ is irreducible in $\text{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.


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$$\binom{x}{n}^k = \left(\frac{x(x-1)\cdots(x-n+1)}{n!} \right)^k$$

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Theorem (R., Windisch; 2021)

$\binom{x}{n}$ is **absolutely** irreducible in $\text{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.

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$$\binom{x}{n}^k = \left(\frac{x(x-1)\cdots(x-n+1)}{n!} \right)^k = (\pm 1) \cdot \left(\pm \binom{x}{n} \right)^k$$

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for certain s

The valuation matrix

$$A_n = (v_p(n+r-i) - v_p(n-i))_{\substack{p \in \mathcal{P}_n, r \in \mathcal{R}_{n,p} \\ 0 \leq i \leq n-1}}$$

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for most n, p :
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Theorem (R., Windisch; 2021)

- $\prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{k_i}$ divides $\binom{x}{n}^m \implies (k_0, k_1, \dots, k_{n-1})^t \in \ker(A_n)$

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$\{p \in \mathcal{P} \mid p \leq n\}$
 (green arrow pointing to $p \in \mathcal{P}_n$)

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- $\prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{k_i}$ divides $\binom{x}{n}^m \implies (k_0, k_1, \dots, k_{n-1})^t \in \ker(A_n)$
- $\text{rank}(A_n) = n - 1 \implies \binom{x}{n}$ absolutely irreducible

The valuation matrix

$$\left(\begin{array}{ccc} * & \dots & * \\ \underbrace{\hspace{10em}}_{n-P} & \underbrace{\begin{array}{c} \text{---} \\ | \\ | \\ | \\ | \\ | \\ \dots \\ | \\ | \\ | \\ | \\ \text{---} \end{array}}_{2P-n} & * \dots * \\ \underbrace{\hspace{10em}}_{n-P} & & \end{array} \right)$$

The diagram illustrates the structure of a valuation matrix. It is a square matrix of size $n \times n$, partitioned into three main blocks. The left and right blocks are square matrices of size $(n-P) \times (n-P)$, each containing a diagonal of asterisks ($*$) and horizontal ellipses (\dots) representing other diagonal elements. The middle block is a rectangular matrix of size $(n-P) \times (2P-n)$, shaded gray, and contains vertical lines representing columns. A red dashed vertical line is positioned to the left of this block. Braces below the matrix indicate the dimensions of these blocks: $n-P$ for the left and right blocks, and $2P-n$ for the middle block.

The valuation matrix

$$\left(\begin{array}{c} * \quad \dots \quad * \quad \left[\begin{array}{c} \text{---} \\ | \\ | \\ | \\ \dots \\ | \\ | \\ | \\ \text{---} \end{array} \right] \quad * \quad \dots \quad * \\ \hline \underbrace{\hspace{10em}}_{n-P} \quad \underbrace{\hspace{10em}}_{2P-n} \quad \underbrace{\hspace{10em}}_{n-P} \end{array} \right)$$

The valuation matrix

$$\left(\begin{array}{ccc} * & \dots & * \\ \hline & & \end{array} \right)$$

The diagram shows a matrix structure with three main sections. The first section on the left is a horizontal bar containing two asterisks (*) separated by a dotted line, with a bracket underneath labeled $n - P$. The middle section is a shaded gray area with a red dashed vertical line on its left edge and a jagged right edge, containing several vertical lines and a central ellipsis (...), with a bracket underneath labeled $2P - n$. The third section on the right is another horizontal bar containing two asterisks (*) separated by a dotted line, with a bracket underneath labeled $n - P$.

$$\left(\begin{array}{ccc} 0 & \begin{array}{c} -1 \\ \vdots \\ -1 \end{array} & \begin{array}{c} 1 \\ \diagdown \dots \diagup \\ 1 \end{array} \\ \hline & & \end{array} \right)$$

The diagram shows a detailed view of a matrix row. It is enclosed in a purple rounded rectangle. The row is divided into three parts: a zero (0) on the left, a vertical column in the middle containing a -1 at the top and bottom with a vertical dotted line in between, and a square block on the right containing a 1 at the top-left and bottom-right corners with a diagonal dotted line connecting them. The entire row is enclosed in large parentheses.

The valuation matrix

$$\left(\begin{array}{c} * \quad \dots \quad * \quad \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right] \quad * \quad \dots \quad * \\ \underbrace{\hspace{10em}}_{n-P} \quad \underbrace{\hspace{10em}}_{2P-n} \quad \underbrace{\hspace{10em}}_{n-P} \end{array} \right)$$

$$\left(\begin{array}{ccc} 0 & \begin{array}{c} -1 \\ \vdots \\ -1 \end{array} & \begin{array}{c} 1 \\ \dots \\ 1 \end{array} \\ & & 0 \end{array} \right)$$

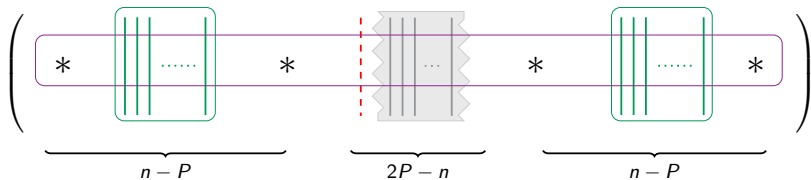
- “inner” rank is $2P - n - 1$
- “inner” and “outer” column spaces have trivial intersection

The valuation matrix

$$\left(\begin{array}{c} * \\ \underbrace{\left(\begin{array}{c} | \quad | \quad | \quad \dots \quad | \\ \hline \end{array} \right)}_{n-P} \\ * \end{array} \right) * \begin{array}{c} \underbrace{\left(\begin{array}{c} | \quad | \quad | \quad \dots \quad | \\ \hline \end{array} \right)}_{2P-n} \\ * \end{array} \left(\begin{array}{c} * \\ \underbrace{\left(\begin{array}{c} | \quad | \quad | \quad \dots \quad | \\ \hline \end{array} \right)}_{n-P} \\ * \end{array} \right)$$

The diagram illustrates the valuation matrix as a product of three matrices. The first matrix is a square matrix of size $n-P$, represented by a green box containing vertical lines and a horizontal ellipsis, with asterisks on either side. The second matrix is a rectangular matrix of size $2P-n$, represented by a gray box with a red dashed vertical line on the left and a jagged right edge, containing vertical lines and a horizontal ellipsis. The third matrix is a square matrix of size $n-P$, identical to the first, with asterisks on either side. The entire expression is enclosed in large parentheses.

The valuation matrix

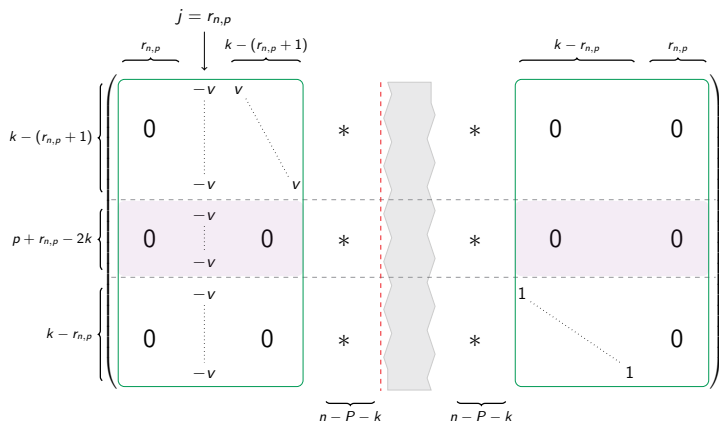


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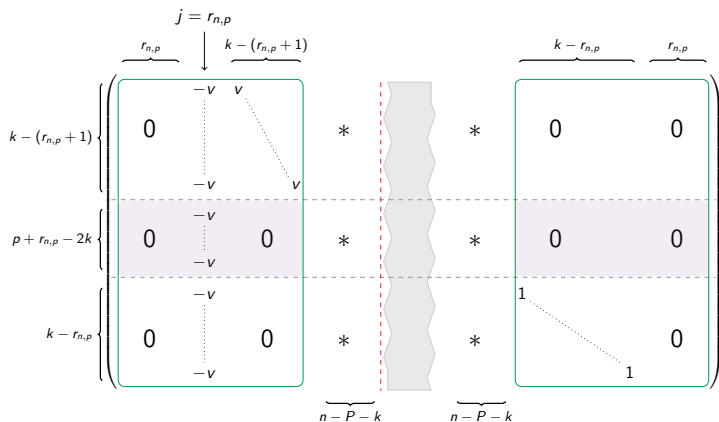
$$\left(\underbrace{\left(* \begin{array}{|c|} \hline \vdots \\ \hline \end{array} * \right)}_{n-P} \underbrace{\left(\begin{array}{|c|} \hline \vdots \\ \hline \end{array} \right)}_{2P-n} \underbrace{\left(\begin{array}{|c|} \hline \vdots \\ \hline \end{array} * \right)}_{n-P} \right)$$

$$\left(\begin{array}{c} \underbrace{\quad}_{r_{n,p}} \quad \downarrow \quad j = r_{n,p} \\ \underbrace{\quad}_{j = p-1} \\ \left(\begin{array}{|c|} \hline 0 \quad \begin{array}{c} -v \\ \vdots \\ -v \end{array} \quad \begin{array}{c} v \\ \vdots \\ v \end{array} \\ \hline \end{array} \right) \quad * \quad \dots \quad * \quad \left(\begin{array}{|c|} \hline 1 \quad \begin{array}{c} \vdots \\ \vdots \\ 1 \end{array} \quad 0 \\ \hline \end{array} \right) \end{array} \right)$$

The "outer" rank



The “outer” rank



Theorem (R., Windisch, 2021)

Let $n > 10$, P the maximal prime with $P \leq n$, and $2 \leq k \leq n - P$.

Then there exists a prime $p > 2k$ with $p \mid n(n-1) \cdots (n-k+1)$.

The binomial polynomials

$$\begin{pmatrix} x \\ n \end{pmatrix} \rightsquigarrow \text{valuation matrix } A_n \in \mathbb{Z}^{m \times n}$$

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$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \in \text{Int}(\mathbb{Z}) \quad \text{abs. irred.}$$



All int.-val. polynomials over DVRs

$(R, pR) \dots$ DVR

$$\text{Int}(R) = \{f \in q(R)[x] \mid f(R) \subseteq R\}$$

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$$f \sim h^r$$

$$g \sim h^s$$



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Ex. in $\text{Int}(\mathbb{Z}_{(3)})$

$$\left(\frac{(x^2 + 9)(x - 5)^3(x - 1)(x - 7)}{3^2} \right)^k$$

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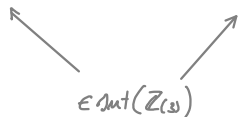
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 $e_{\text{int}}(Z_{(3)})$

Theorem (Hieber, Nakato, R.; 2023)

Let (R, pR) be a DVR and $f \in R[x]$ s.t. $F = \frac{f}{p^n}$ irreducible.

TFAE:

- F is absolutely irreducible

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- F^S factors uniquely for $S = 2(n+1)n^{q \lceil \frac{n}{2} \rceil}$ where $q = |R/pR|$

Factors of F^j

$$\left(\frac{\prod_g g^{m_g}}{p^n} \right)^j = F_1 \cdot F_2$$

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The diagram shows the factorization of the j -th power of a fraction. The fraction $\left(\frac{\prod_g g^{m_g}}{p^n} \right)^j$ is shown with a blue box around the numerator and a blue arrow labeled f pointing to it. This is equal to the product of two factors, F_1 and F_2 . The factor F_1 is approximately $\frac{\prod_g g^{k_g}}{p^\ell}$, with a green box around the numerator and a green arrow labeled f_1 pointing to it. The factor F_2 is approximately $\frac{\prod_g g^{jm_g - k_g}}{p^{jn - \ell}}$, with a green box around the numerator and a green arrow labeled f_2 pointing to it. A large green bracket on the right groups F_1 and F_2 together, with the label $\in \text{Int}(\mathcal{R})$.

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$$\sum_g v(g(w)) \left(k_g - \frac{\ell}{n} m_g \right) = 0$$

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$$(k_g)_g \in \frac{\ell}{n} m_g + \text{fd-ker}(f)$$

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↙ f (pointing to the boxed fraction)

↙ f_1 (pointing to the top fraction)

↙ f_2 (pointing to the bottom fraction)

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The fd-kernel

fdp-matrix: matrix A with $\text{fd-ker}(f) = \ker(A)$

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Lemma

For all $a \in R$ with $v(f(a)) = n$ exists a $w \in W$ with

$$(v(g(a)))_{g \in P} = (v(g(w)))_{g \in P}$$

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Construction of non-unique factorizations

$$0 \neq (v_g)_g \in \text{fd-ker}(f) \implies F^{k+\ell} = \frac{\prod_g g^{km_g - v_g}}{p^{kn}} \cdot \frac{\prod_g g^{\ell m_g + v_g}}{p^{\ell n}}$$

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large enough ↓

$-v_g \in \text{fd-ker}(f)$ ↓

$\in \text{Aut}(R)$

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Bounding $k + \ell$

$$k + \ell \geq (n + 1) \left(\left[\max \left\{ \frac{v_g}{m_g} : v_g > 0 \right\} \right] + \left[\max \left\{ \frac{-v_g}{m_g} : v_g < 0 \right\} \right] \right)$$

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$$\min \{ \|\mathbf{v}\|_\infty : 0 \neq \mathbf{v} \in \text{fd-ker}(f) \cap \mathbb{Z}^P \}$$

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- How many different factorizations? Of which lengths?
- Construct factorizations of powers of irreducibles

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