On transfer Krull orders in global fields

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Ring Theory Seminar

November 30, 2023

Outline

1. Global fields and orders
2. Krull monoids and the arithmetic of $\mathcal{O}$
3. Previous results
4. Transfer Krull orders

Algebraic number fields and rings of integers

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Let F be a field. An *algebraic function field* K/F of one variable over F is an extension field $F \subseteq K$, such that K is a finite extension of $F(x)$, where $x \in K$ is transcendental over F.

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Holomorphy domains

- Let $K/\mathbb{F}_q$ be an algebraic function field over the finite field with $q = p^n$ elements, p a prime. A *prime divisor* or a *place* $\mathbf{p}$ of K is a maximal ideal of a discrete valuation domain $\mathcal{O}_{\mathbf{p}}$, such that $\mathrm{q}(\mathcal{O}_{\mathbf{p}}) = K$.

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Definition

Let $\emptyset \neq S \subsetneq \mathcal{P}(K)$ be a finite set of prime divisors. The *holomorphy domain* of S is the ring

$$\mathcal{O}_{K,S} = \bigcap_{\mathbf{p}\in\mathcal{P}(K)\setminus S} \mathcal{O}_{\mathbf{p}}.$$

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- An *order* in the holomorphy domain $\mathcal{O}_{K,S}$ is a subring $\mathcal{O} \subsetneq \mathcal{O}_{K,S}$ such that $\mathsf{q}(\mathcal{O}) = \mathsf{q}(\mathcal{O}_{K,S}) = K$ and $\overline{\mathcal{O}} = \mathcal{O}_{K,S}$.

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Theorem (Artin-Whaples, 1945)

Let K be a field. Then K is a global field if and only if the *product formula*

$$\prod_{\mathrm{v}} |x|_{\mathrm{v}} = 1$$

is satisfied for every $x \in K^{\times}$, where v ranges over all equivalence classes of multiplicative valuations on K.

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- Generalized Riemann Hypothesis for number fields and Weil conjectures for function fields (all proven).
- Global class field theory and Chebotarevs density theorem.

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Let $\mathbb{F}_q$ be a finite field. For every finite abelian group G, there exists some function field $K/\mathbb{F}_q$ and some finite set $\emptyset \neq S \subsetneq \mathcal{P}(K)$ such that $G \cong \mathrm{Cl}(\mathcal{O}_{K,S})$.

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- The analogous statement for number fields is still an open problem.

Basic facts about orders

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- The *conductor* of $\mathcal{O}$

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- The finitely many $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ with $\mathfrak{p} \supseteq \mathfrak{f}$ are called *irregular* prime ideals.

The Picard group of an order

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- We have $\mathrm{Pic}(\mathcal{O}_{K,S}) = \mathrm{Cl}(\mathcal{O}_{K,S})$.
- $\mathrm{Pic}(\mathcal{O})$ is finite and every class contains infinitely many prime ideals.

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- H is *half-factorial* if for every nonunit $x \in H$, we have $|\mathsf{L}(x)| = 1$.

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- Let $x \in H$. The *elasticity* of x is defined as

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Definition

H is called a *Krull monoid* if there is a divisor homomorphism $\varphi : H \to D$, where D is factorial, such that for every $a \in D$, there is a finite subset $\emptyset \neq X \subset H$ with $a = \gcd(\varphi(X))$. The *class group* of H is $\mathcal{C}(H) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$.

Transfer homomorphisms

Definition

Let H and B be monoids. A homomorphism $\phi : H \to B$ is called a *transfer homomorphism* if it has the following two properties:

(1) $B = \phi(H) \cdot B^{\times}$ and $\phi^{-1}(B^{\times}) = H^{\times}$

(2) If $u \in H$, $b, c \in B$ and $\phi(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\phi(v) \simeq b$ and $\phi(w) \simeq c$.

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- Transfer homomorphisms preserve the arithmetic, in particular they preserve sets of lengths.

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Theorem

Let H be a Krull monoid with class group $\mathcal{C}(H)$ such that every class contains a prime divisor. Then there is a transfer homomorphism $H \to \mathcal{B}(\mathcal{C}(H))$.

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Let G be an abelian group. Then $\mathcal{B}(G)$ is half-factorial if and only if $|G| \leq 2$.

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Theorem (Carlitz, 1960)

$\mathcal{O}_K$ is half-factorial if and only if $|\mathrm{Cl}(\mathcal{O}_K)| \leq 2$.

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Example

Every half-factorial monoid is transfer Krull. Indeed, let H be half-factorial. Then $\ell : H \to (\mathbb{N}_0, +)$, where $\ell(x) \in \mathsf{L}(x)$ is the unique factorization length of x, is a transfer homomorphism.

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- For which orders $\mathcal{O}$ is there a transfer homomorphism $\mathcal{O}^\bullet \to \mathcal{O}_{K,S}^\bullet$?
- For which orders $\mathcal{O}$ is $\mathcal{O}^\bullet \hookrightarrow \mathcal{O}_{K,S}^\bullet$ a transfer homomorphism?

T-block monoids and the arithmetic of $\mathcal{O}$

- Let G be an abelian group, T a monoid and $\iota : T \to G$ a homomorphism. Then

$$\mathcal{B}(G, T, \iota) = \{St \in \mathcal{F}(G) \times T : \sigma(S) + \iota(t) = 0\}$$

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Theorem

Let $\mathcal{O}$ be an order in a global field K. There is a transfer homomorphism $\mathcal{O}^\bullet \to \mathcal{B}(G, T, \iota)$, where $G = \mathrm{Pic}(\mathcal{O})$,

$$T = \prod_{\mathfrak{p} \supseteq \mathfrak{f}} \mathcal{O}^\bullet_{\mathfrak{p}\,\mathrm{red}}$$

and $\iota : T \to G$ is the canonical homomorphism.

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- We call an order $\mathcal{O}$ *locally half-factorial* if $\mathcal{O}_{\mathfrak{p}}$ is half-factorial for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$.
- Is every half-factorial order locally half-factorial?

The elasticity

Theorem (Halter-Koch, 1995)

Let $\mathcal{O}$ be an order in a global field K. Then $\rho(\mathcal{O}) < \infty$ if and only if the map

$$\mathrm{Spec}(\mathcal{O}_{K,S}) \to \mathrm{Spec}(\mathcal{O})$$

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Let $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ with $\mathfrak{P}_1, \ldots, \mathfrak{P}_s \in \mathrm{Spec}(\mathcal{O}_K)$ lying over $\mathfrak{p}$ and $s \geq 2$. Let p be a prime element of $\overline{\mathcal{O}_\mathfrak{p}}$. Then $|\mathrm{v}_p(\mathcal{A}(\mathcal{O}_\mathfrak{p}))| = \infty$. A product of few atoms of high valuation can have long factorizations with atoms of small valuation. On the other hand, if $s = 1$, then $\mathrm{v}_p(\mathcal{A}(\mathcal{O}_\mathfrak{p}))$ is finite.

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Corollary

If $\mathcal{O}$ is half-factorial, then the map $\mathfrak{P} \mapsto \mathfrak{P} \cap \mathcal{O}$ is bijective.

Orders in quadratic number fields

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- Every conductor ideal $\mathfrak{f}$ is of the form $\mathfrak{f} = f\mathcal{O}_K$ for some $f \in \mathbb{N}_{\geq 2}$.
- The only order with conductor f is the minimal order $\mathbb{Z} + f\mathcal{O}_K$.

Orders in quadratic number fields

Theorem (Halter-Koch, 1983)

Let K be a quadratic number field with ring of integers $\mathcal{O}_K$ and let $\mathcal{O}$ be an order in K with conductor $f \in \mathbb{N}_{\geq 2}$. Then $\mathcal{O}$ is half-factorial if and only if the following conditions are satisfied.

(i) $\mathcal{O}_K$ is half-factorial.

(ii) $\mathcal{O} \cdot \mathcal{O}_K^\times = \mathcal{O}_K$.

(iii) f is either a prime or twice an odd prime.

If this is the case, then $\mathcal{O}$ is locally half-factorial.

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- The only half-factorial imaginary quadratic order is $\mathbb{Z}[\sqrt{-3}]$ with conductor 2.

Seminormal orders

- Let R be a noetherian domain. We call R *seminormal* if for all $x \in \overline{R} \setminus R$, there are infintely many $n \in \mathbb{N}$ with $x^n \notin R$.

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Lemma

An order $\mathcal{O}$ is seminormal if and only if $\mathfrak{f}$ is squarefree if and only if $\mathfrak{f}$ is a radical ideal in $\mathcal{O}_{K,S}$.

Seminormal orders

Theorem (Geroldinger-Kainrath-Reinhart, 2015)

Let $\mathcal{O}$ be a seminormal order in a global field K. Then $\mathcal{O}$ is half-factorial if and only if the following conditions are satisfied.

(i) $\mathcal{O}_{K,S}$ is half-factorial.

(ii) The map

$$\mathrm{Spec}(\mathcal{O}_{K,S}) \to \mathrm{Spec}(\mathcal{O}),$$
$$\mathfrak{P} \mapsto \mathfrak{P} \cap \mathcal{O}$$

is bijective.

(iii) $|\,\mathrm{Pic}(\mathcal{O})| = |\,\mathrm{Cl}(\mathcal{O}_{K,S})|$.

If this is the case, then $\mathcal{O}$ is locally half-factorial.

Half-factoriality of $\mathcal{O}$

Theorem (R., 2023)

Let $\mathcal{O}$ be an order in a global field K with conductor $\mathfrak{f} = \mathfrak{P}_1^{k_1} \ldots \mathfrak{P}_s^{k_s}$ and let $\mathfrak{p}_i = \mathfrak{P}_i \cap \mathcal{O}$. Then $\mathcal{O}$ is half-factorial if and only if the following conditions are satisfied.

(i) $\mathcal{O}_{K,S}$ is half-factorial.

(ii) $\mathcal{O} \cdot \mathcal{O}_{K,S}^{\times} = \mathcal{O}_{K,S}$.

(iii) For all $i \in [1, s]$, we have $k_i \leq 4$ and $\mathsf{v}_{p_i}(\mathcal{A}(\mathcal{O}_{\mathfrak{p}_i})) \subseteq \{1, 2\}$, where p_i is an arbitrary prime element of $\overline{\mathcal{O}}_{\mathfrak{p}_i}$. If $\mathfrak{P}_i$ is principal, we have $k_i \leq 2$ and $\mathsf{v}_{p_i}(\mathcal{A}(\mathcal{O}_{\mathfrak{p}_i})) = \{1\}$.

Half-factorial $\not\Longrightarrow$ locally half-factorial

- Let $K = \mathbb{Q}(\omega)$, where ω is a root of

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- Then $\mathcal{O}$ is half-factorial and we have $\mathsf{v}_p(\mathcal{A}(\mathcal{O}_\mathfrak{p})) = \{1, 2\}$, where $\mathfrak{p} = \mathfrak{P}_1 \cap \mathcal{O}$.

Transfer Krull orders

Theorem (R., 2023)

Let $\mathcal{O}$ be a non-half-factorial order in a global field K. Let $\mathfrak{f}$ be the conductor of $\mathcal{O}$.

Then $\mathcal{O}^\bullet$ is transfer Krull if and only if $\mathcal{O} \cdot \mathcal{O}_{K,S}^\times = \mathcal{O}_{K,S}$ and $\mathcal{O}$ is locally half-factorial.

In this case, $\mathfrak{f}$ is cubefree and $\mathcal{O}^\bullet \hookrightarrow \mathcal{O}_{K,S}^\bullet$ is a transfer homomorphism.

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- Transfer Krull orders are indeed close to $\mathcal{O}_{K,S}$ algebraically.

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- Transfer Krull orders are indeed close to $\mathcal{O}_{K,S}$ algebraically.
- The previous example shows that the characterizations of half-factorial orders and non-half-factorial transfer Krull orders are different.

Quadratic and seminormal orders revisited

Corollary

An order $\mathcal{O}$ in a quadratic number field K with conductor $f \in \mathbb{N}_{\geq 2}$ is transfer Krull if and only if the following two conditions are satisfied.

(i) $\mathcal{O} \cdot \mathcal{O}_K^\times = \mathcal{O}_K$.

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Corollary

A seminormal order $\mathcal{O}$ in a global field K is transfer Krull if and only if the following two conditions are satisfied.

(i) The map $\mathfrak{P} \mapsto \mathfrak{P} \cap \mathcal{O}$ is bijective.

(ii) $|\,\mathrm{Pic}(\mathcal{O})| = |\,\mathrm{Cl}(\mathcal{O}_{K,S})|$.

Thank you for your attention!