Every finitely generated abelian group is the class group of a generalized cluster algebra

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Triangulations

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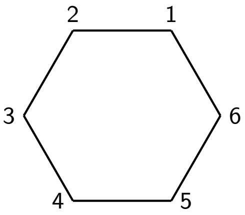

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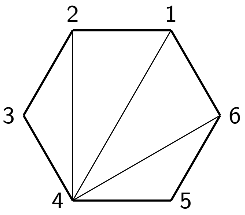

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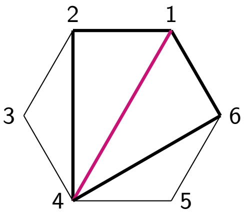

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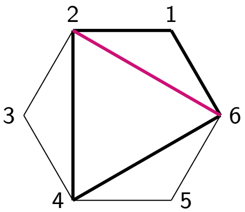

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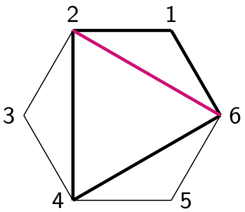

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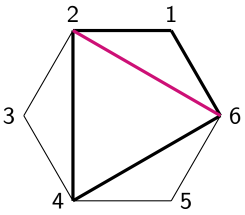

FLIP

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Knowing the values of a triangulation, we can know all the other values

The cluster algebra of the n+3-gon

Define the commutative ring A_n generated by the variables

$$\{a_{ij} \mid \forall i, j \text{ with } 1 \leq i < j \leq n\}$$

and with relations generated by

$$\{a_{ik}a_{jl} = a_{ij}a_{kl} + a_{il}a_{jk} \mid \forall i, j, k, l \text{ with } i < j < k < l\}.$$

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Cluster algebras: an idea

A cluster algebra is a commutative ring $\mathcal{A}$ with the following data

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A cluster algebra is a commutative ring $\mathcal{A}$ with the following data

- A special set of generators (the cluster variables).
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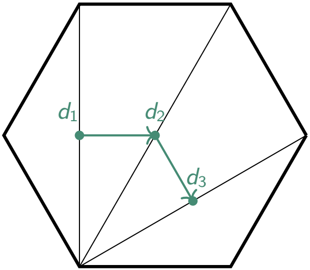

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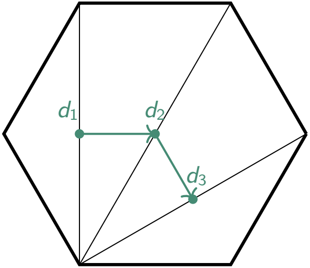

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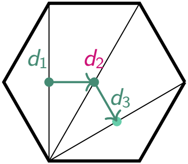
The matrix B is antisymmetric!

How does the matrix change under flips?



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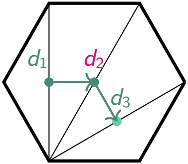

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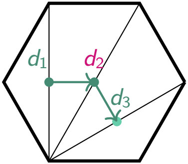

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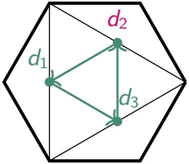

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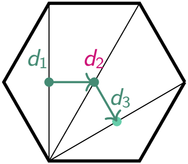

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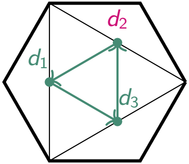

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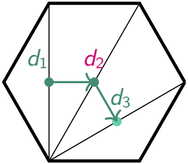

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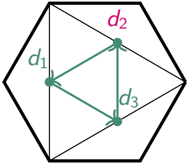

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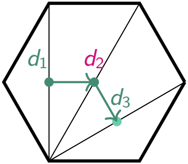

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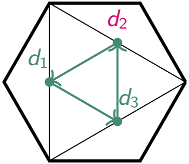

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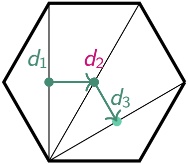

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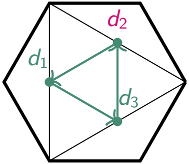

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For each column i of B, fix a positive integer $d_i \in \mathbb{N}$, such that $d_i \mid b_{ji}$ for every $j \in \{1, \ldots, n\}$. We denote by β_{ji} the integer b_{ji}/d_i.

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- $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a cluster, i.e. a set of algebraically independent indeterminates over R.

Mutation of a seed

Let $(\mathbf{x}, \rho, B)$ be a generalized seed.[1] We want to mutate the seed in direction i, i.e. to build another seed $(\mathbf{x}', \rho', B')$.

[1] For $x \in \mathbb{R}$, define the function $[x]_+ := x$ if $x \geq 0$, and $[x]_+ = 0$ otherwise.

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$$\mathbf{x}' = (\mathbf{x} \setminus x_i) \cup \{x_i'\}$$

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The polynomials $f_i := x_i x_i' \in R[\mathbf{x}]$ are called EXCHANGE POLYNOMIALS.

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The GENERALIZED CLUSTER ALGEBRA $\mathcal{A} = \mathcal{A}(\mathbf{x}, \rho, B)$ is the subalgebra of the rational functions $R(x_1, \ldots, x_n)$ generated by all the cluster variables.

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And in this case we say that $\mathcal{A}(\mathbf{x}, B)$ is a CLUSTER ALGEBRA.

An example

- Let $\mathbf{x} = \{x_1, x_2\}$, $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, $d_1 = 1, d_2 = 2$, and $\rho_1 = \{1, 1\}, \rho_2 = \{1, 2, 1\}$.

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$$x_1 x_1' = x_2 + 1$$

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- Let $\mathbf{x} = \{x_1, x_2\}$, $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, $d_1 = 1, d_2 = 2$, and $\rho_1 = \{1,1\}, \rho_2 = \{1,2,1\}$.
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$$x_2 x_2' = \sum_{j=0}^{d_2} \rho_{2j} \prod_{k=1}^{2} x_k^{j[\beta_{k2}]_+ + (d_2 - j)[-\beta_{21}]_+}$$

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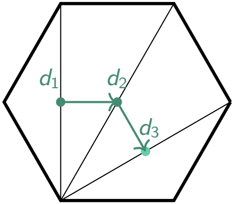
- $\mathcal{A}(\mathbf{x}, \rho, B) = R[x_1, x_2, x_3, x_4, x_5, x_6]$, with $x_{k-1}x_{k+1} = \begin{cases} 1 + x_k & k \in 2\mathbb{Z} \\ 1 + 2x_k + x_k^2 & \text{otherwise} \end{cases}$

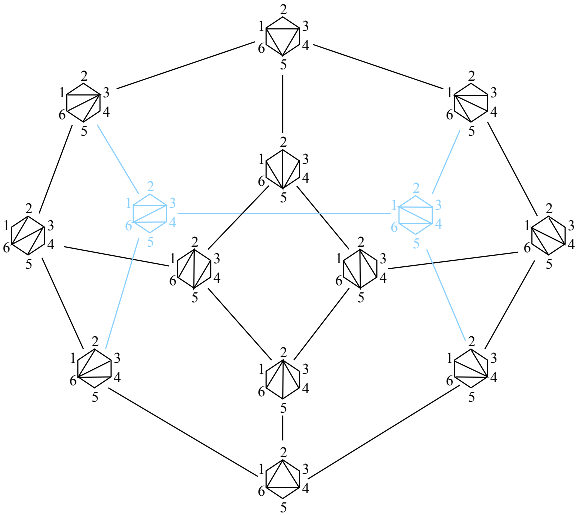
Our old example A_n

Let's consider again triangulations of our hexagon P_6...



$$B(T) = (b_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Our old example A_n



From L.K. Williams, *Cluster algebras: An introduction, 2012*

.

Our old example A_n

$$A_n = \mathcal{A}(\mathbf{x}, B), \text{ with } \mathbf{x} = \{x_1, \dots, x_n\}, \text{ and } b_{ij} = \begin{cases} -1 & \text{if } j = i+1 \\ 1 & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

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$$A_3 = R\left[x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{1+x_2}{x_3}, \frac{x_1+(1+x_2)x_3}{x_1x_2}, \frac{(1+x_2)x_1+x_3}{x_2x_3}, \frac{(1+x_2)(x_1+x_3)}{x_1x_2x_3}\right]$$

Some properties

	Cluster Algebras	Generalized Cluster Algebras
$\mathcal{A} \subseteq R[\mathbf{x}^{\pm 1}]$ (Laurent phenomenon)	✓	✓
$\mathcal{A}^\times = R^\times$	✓	✓
cluster variables are strong atoms	✓	✓
exchange polynomials have positive coefficients	✓	✓
full finite type classification	✓	✓
FF-domains	✓	✓
Class groups	$\mathbb{Z}^r$	$\mathbb{Z}^r/I$

Table of Contents

Krull domains and their class groups

- Krull domains are a higher dimensional generalization of Dedekind domains [2]

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- The principal ideal xA can be written uniquely as

$$xA = {\mathfrak{p}_1}^{a_1} \cdot_v \cdots \cdot_v {\mathfrak{p}_t}^{a_t},$$

with $a_i \in \mathbb{N}_0$.

[2] integral domains where every non-zero ideals factors uniquely into prime ideals.

Realization theorems

- Claborn 1966: every abelian group is the class group of a Dedekind domain.
- Leedham-Green 1972: every abelian group is the class group of a Dedekind domain that is the quadratic extension of a principal ideal domain.
- Rosen 1976: every countable abelian group is the class group of an elliptic Dedekind domain.
- Smertnig 2017: every abelian group is the class group of a simple Dedekind domain.
- Still open: is every finite abelian group isomorphic the class group of the ring of integers of a number field?

Are cluster algebras Krull domains?

- FINITE TYPE CLUSTER ALGEBRAS[3] are Krull domains

[3]Cluster algebras with finitely many cluster variables

[4]The matrix B is non-singular.

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Class group of a cluster algebra

Theorem (Garcia Elsener, Lampe, Smertnig 2019)

Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, B)$ be a cluster algebra. Assume that $\mathcal{A}$ is a Krull domain. Then the class group $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$ is

$$\mathcal{C}(\mathcal{A}) \cong \mathbb{Z}^r,$$

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- for acyclic cluster algebras (Garcia Elsener, Lampe, Smertnig 2019)
- full rank upper cluster algebras (P. 2023)

Theorem (P. 2024)

Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, \rho, B)$ be a generalized cluster algebra. Assume that $\mathcal{A}$ is a Krull domain. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the pairwise distinct height-1 prime ideals of $\mathcal{A}$ containining one of $x_1, \dots, x_n$. Suppose that

$$x_i\mathcal{A} = \mathfrak{p}_1^{a_{i1}} \cdot_v \cdots \cdot_v \mathfrak{p}_r^{a_{ir}},$$

with $\mathbf{a}_i = (a_{ij})_{j=1}^r \in \mathbb{N}_0^r$. Then

$$\mathcal{C}(\mathcal{A}) \cong \mathbb{Z}^r / \langle \mathbf{a}_i \mid i \in [1, n] \rangle$$

and it is generated by $[\mathfrak{p}_1], \dots, [\mathfrak{p}_r]$.

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What can we do with our class group?

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- If G is infinite, then for every non-empty finite subset $L = \{l_1, \ldots, l_k\}$ of $\mathbb{N}_{\geq 2}$ there exists $a \in \mathcal{A}$ such that

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  where u_{ij} are atoms of $\mathcal{A}$.
- If G is finite, then there are arithmetic invariants that make us completely understand the algebraic structure of $\mathcal{A}$.

Height-1 prime ideals

Proposition (P. 2024)

Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, \rho, B)$ be a generalized cluster algebra. Suppose that $\mathcal{A}$ is acyclic and full rank. Let $r_1, \ldots, r_t \in R[\mathbf{x}]$ be the distinct irreducible factors of the exchange polynomial $f_i = x_i x_i'$. Then

$$\{r_j A_i \cap \mathcal{A} \mid j \in [1, t]\}$$

is the set of all the height-1 prime ideals of $\mathcal{A}$ that contain x_i.

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Here A_i denote the Laurent polynomial ring $R[x_1^{\pm 1}, \ldots, x_i'^{\pm 1}, \ldots, x_n^{\pm 1}]$.

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Therefore $\mathcal{C}(\mathcal{A}) \cong \mathbb{Z}^2 / \langle \mathbf{a}_1, \mathbf{a}_2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

A realization theorem

Theorem (P. 2024)

Let G be a finitely generated abelian group. Then there exists an acyclic and coprime generalized cluster algebra $\mathcal{A}$ over an algebraically closed field k such that $\mathcal{A}$ is a Krull domain, its class group $\mathcal{C}(\mathcal{A})$ is isomorphic to G and each class of $\mathcal{C}(\mathcal{A})$ contains exactly $|k|$ prime divisors.

Idea of the construction

Let G be $\mathbb{Z}^4 \times \mathbb{Z}/3\mathbb{Z}$.

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Idea of the construction

Let G be $\mathbb{Z}^4 \times \mathbb{Z}/3\mathbb{Z}$.

- Let $\mathbf{x} = \{x_1, x_2, x_3, x_4\}$.
- Let $B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}$, $\rho_1 = \{1, 1\}$ and $\rho_2 = \{1, 3, 3, 1\}$, $\rho_3 = \rho_4 = \{1, 1\}$
- $f_1 = x_4^5 + 1$, $f_2 = (x_3 + 1)^3$, $f_3 = x_2 + 1$, $f_4 = x_1 + 1$.Write: $f_1 = r_1 \cdots r_5$, and $f_2 = g_2^3$.
- $\{\underbrace{r_1 A_1 \cap \mathcal{A}, r_2 A_1 \cap \mathcal{A}, r_3 A_1 \cap \mathcal{A}, r_4 A_1 \cap \mathcal{A}, r_5 A_1 \cap \mathcal{A}}_{\mathfrak{p}_{11}, \dots, \mathfrak{p}_{15} \ni x_1}, \underbrace{g_2 A_2 \cap \mathcal{A}}_{=: \mathfrak{p}_2 \ni x_2}, \underbrace{f_3 A_3 \cap \mathcal{A}}_{=: \mathfrak{p}_3 \ni x_3}, \underbrace{f_4 A_4 \cap \mathcal{A}}_{=: \mathfrak{p}_4 \ni x_4}\}$.
- $x_1 \mathcal{A} = \mathfrak{p}_{11} \cdot_v \mathfrak{p}_{12} \cdot_v \mathfrak{p}_{13} \cdot_v \mathfrak{p}_{14} \cdot_v \mathfrak{p}_{15}$, $x_2 \mathcal{A} = \mathfrak{p}_2^3$, $x_3 \mathcal{A} = \mathfrak{p}_3$, $x_4 \mathcal{A} = \mathfrak{p}_4$
- $\mathcal{C}(\mathcal{A}(\mathbf{x}, \rho, B)) \cong \mathbb{Z}^8 / \langle \mathbf{a}_1, \dots, \mathbf{a}_5 \rangle \cong \mathbb{Z}^4 \times \mathbb{Z}/3\mathbb{Z}$.

Summary

- The class groups of generalized cluster and cluster algebras (that are Krull domains) are always finitely generated.
- Cluster algebras does not have torsion, while generalised cluster algebra may have torsion.
- It is possible to realize every finitely generated abelian group as class group of a generalized cluster algebra.

Summary

- The class groups of generalized cluster and cluster algebras (that are Krull domains) are always finitely generated.
- Cluster algebras does not have torsion, while generalised cluster algebra may have torsion.
- It is possible to realize every finitely generated abelian group as class group of a generalized cluster algebra.

Open questions:

- Is there a characterization of (generalized) cluster algebras that are Krull domains?
- Can we say something more about the arithmetic of generalized cluster algebras?

Thank you for your attention!