Integer-valued polynomials over upper triangular matrix rings

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Outline

Notations

Throughout the talk, unless otherwise stated, all the considered rings will be commutative rings with identity, each ring and its subrings have the same identity. Let R be a ring and E be a subset of R. Then $M_n(E)$ and $T_n(E)$ denote the set of $n \times n$ matrices and upper triangular matrices with entries from E,

Introduction

Definitions

If D is an integral domain and K is the field of fractions of D, the ring of integer-valued polynomials over D is defined by

$$\mathrm{Int}(D) := \{f \in K[x] | f(D) \subseteq D\}.$$

The first systematic study of the algebraic properties of $\mathrm{Int}(D)$ was done in consecutive 1919 papers of Ostrowski [3] and Polya [4] of the same title. For literature on $\mathrm{Int}(D)$ the reader is referred to Cahen and Chabert [1].

Introduction

Definitions

Let R be a ring and $f(x)$ and $g(x)$ be two elements of $R[x]$. Then $(fg)(x)$ denotes the product of $f(x)$ and $g(x)$ in $R[x]$. If R is not a commutative ring and $\alpha \in R$, then $(fg)(\alpha)$ need not equal $f(\alpha)g(\alpha)$. In this case, if $f(x) = \sum_{i=1}^{n} a_i x^i$, then we may express

$$(fg)(x) := \sum_{i=1}^{n} a_i g(x) x^i \qquad (*)$$

Let R_1 and R_2 be (not necessarily commutative) rings such that $R_1 \subseteq R_2$. The set of integer-valued polynomials over R_1 with coefficients in R_2 is:

$$\mathrm{Int}_{R_2}(R_1) = \{f \in R_2[X] | f(R_1) \subseteq R_1\}.$$

Introduction

History for noncommutative case

In 2012, Werner [6] showed:

Theorem

Let D be an integral domain.Then the set

$$\mathrm{Int}(M_n(D)) := \mathrm{Int}_{M_n(K)}(M_n(D)),$$

where K is the field of fractions of D, with ordinary addition and multiplication $(*)$ is a ring

This theorem follows from the fact that there is a set of invertible matrices that forms a basis for $M_n(D)$ as a D-algebra.

History for noncommutative case

Example

$$\mathcal{U} = \{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\}$$

is a basis of invertible matrices for $M_2(D)$ as a D-algebra.

For $n \geq 3$, one may take the $n^2 - n$ elementary matrices that differ from In by having a 1 in exactly one off-diagonal entry; the identity matrix I_n; and $n-1$ permutation matrices, each with a single distinct fixed point. Alternatively, a different construction for $n > 2$ is given in:
[Lopez-Permouth et al., Algebras having bases consisting entirely of units. Groups, Rings, and Group Rings, Contem. Math., 499: 219–228, 2009].

History for noncommutative case

Note

In general, there is not a set of invertible matrices that forms a basis for $T_n(D)$ as a D-algebra.

History for noncommutative case

Example 1

The set of unit elements of $T_2(\mathbb{Z}_2)$ is

$$\mathcal{U} = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\}.$$

So $T_2(\mathbb{Z}_2)$ can not generated by units.

Example 2

The set of unit elements of $T_2(\mathbb{Z})$ is

$$\mathcal{U} = \{\begin{bmatrix} \pm 1 & a \\ 0 & \pm 1 \end{bmatrix}, |a \in \mathbb{Z}\}.$$

So $T_2(\mathbb{Z})$ can not generated by units.

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History for noncommutative case

Proof of Example 2

Let

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} u_1 & a_1 \\ 0 & v_1 \end{bmatrix} + c_2 \begin{bmatrix} u_2 & a_2 \\ 0 & v_2 \end{bmatrix} + \cdots + c_m \begin{bmatrix} u_m & a_m \\ 0 & v_m \end{bmatrix}.$$

So

$$1 = c_1u_1 + c_2u_2 + \cdots + c_mu_m,$$

$$0 = c_1v_1 + c_2v_2 + \cdots + c_mv_m.$$

Hence

$$1 = c_1(u_1 - v_1) + c_2(u_2 - v_2) + \cdots + c_m(u_m - v_m),$$

which is a contradiction, since $(u_i - v_i) = 0$ or $(u_i - v_i) = \pm 2$.

Introduction

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History for noncommutative case

In 2017, Frisch ([2]) proved that:

Theorem

Let D be an integral domain. Then

$$\mathrm{Int}(T_n(D)) := \mathrm{Int}_{T_n(K)}(T_n(D)),$$

where K is the field of fractions of D, with ordinary addition and multiplication $(*)$ is a ring is a ring.

For the another proof, see Sedighi et al., [5].
The aim of this talk is introduce and study the equalizing of $\mathrm{Int}(M_n(D))$ and $\mathrm{Int}(T_n(D))$.

Outline

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Ideals of $\mathrm{Int}(M_n(D))$

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Notation

Let $\mathcal{U} = \{U_1, U_2, \ldots, U_{n^2}\}$ be a set of invertible matrices that forms a basis for $M_n(D)$ as a D-algebra. Let $<\mathcal{U}>$ be the multiplicative subgroup of $M_n(D)$ generated by U. For any $A \in M_n(D)$, define $\mathcal{U}(A) = \{UAU^{-1} | U \in <\mathcal{U}>\}$.

Definition

Let $\mathfrak{a}$ be an ideal of D and let $A \in M_n(D)$. Werner defined $\mathcal{J}_{M_n(\mathfrak{a}),A} := \{f \in \mathrm{Int}(M_n(D)) | f(B) \in M_n(\mathfrak{a})$ for all $B \in \mathcal{U}(A)\}$.

We **recall** that any ideal J of $M_n(D)$ has the form $M_n(\mathfrak{a})$ for a uniquely determined ideal $\mathfrak{a}$ of R.

In 2012, Werner [6] showed:

Theorem

Let $\mathfrak{a}$ be an ideal of D and let $A \in M_n(D)$. Then $\mathcal{J}_{M_n(\mathfrak{a}),A}$ is an ideal of $\mathrm{Int}(M_n(D))$

Again, this theorem follows from the fact that there is a set of invertible matrices that forms a basis for $M_n(D)$ as a D-algebra.

For more information about the properties the ideal $\mathcal{J}_{M_n(\mathfrak{a}),A}$ see Werner [6].

Outline

Integer-valued polynomials over upper triangular matrix rings

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Left Ideals of $\mathrm{Int}(T_n(D))$

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Similar to the commutative case:

Definition

Let J be an ideal of $T_n(D)$ and let $A \in T_n(D)$. We define $\mathcal{J}_{J,A} := \{f \in \mathrm{Int}(T_n(D)) | f(A) \in J\}$.

Note

Let R be a commutative ring with identity and let I be an ideal of $T_n(R)$. Then, we have

$$I = \begin{bmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ 0 & I_{22} & \dots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I_{nn} \end{bmatrix}, \quad (1)$$

Left Ideals of $\mathrm{Int}(T_n(D))$

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Note

where

$$I_{ii} \subseteq I_{i(i+1)} \subseteq \cdots \subseteq I_{in}$$

and

$$I_{ii} \subseteq I_{(i-1)i} \subseteq \cdots \subseteq I_{1i}$$

for all $1 \leq i \leq n$.

Theorem

Let J be an ideal of $T_n(D)$ and let $A \in T_n(D)$. Then $\mathcal{J}_{J,A}$ is a left ideal of $\mathrm{Int}(T_n(D))$.

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Left Ideals of $\mathrm{Int}(T_n(D))$

The following example shows that the left ideal $\mathcal{J}_{I,A}$ in the above theorem may not be a right ideal.

Example

Let D be an integral domain and K is the field of fractions of D. Let $\mathfrak{a}$ be a proper ideal of D,

$$I = \begin{bmatrix} \mathfrak{a} & \mathfrak{a} \\ 0 & \mathfrak{a} \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$f(X) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X \text{ and } g(X) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $f(A) = 0$ and hence
$f \in J_{I,A} = \{f \in \mathrm{Int}(T_2(D)) | f(A) \in I\}$.

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Left Ideals of $\mathrm{Int}(T_n(D))$

Example

We have

$$(fg)(X) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

So $(fg)(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin I$. Hence $\mathcal{J}_{I,A}$ is not a right ideal of $\mathrm{Int}(T_2(D))$.

Outline

Equalizing Ideals of $\mathrm{Int}(M_n(D))$

Definition

Let D be a domain. If $\mathfrak{a}$ is an ideal of D and $\mathcal{U} = \{U_1, \ldots, U_{n^2}\}$ is a set of invertible matrices that forms a basis for $M_n(D)$ as a $D-$algebra, then we set

$$\mathfrak{q}_{M_n(\mathfrak{a})} = \{A \in M_n(D) | f(UAU^{-1}) - f(0) \in M_n(\mathfrak{a}) \\ \forall U \in < \mathcal{U} >, \forall f \in \mathrm{Int}(M_n(D))\}.$$

$\mathfrak{q}_{M_n(\mathfrak{a})}$ is called the equalizing ideal of $M_n(\mathfrak{a})$ in $\mathrm{Int}(M_n(D))$.

Equalizing Ideals of $\mathrm{Int}(M_n(D))$

Proposition

Let $\mathfrak{a}$ be an ideal of D and $\mathcal{U} = \{U_1, \ldots, U_{n^2}\}$ be a set of invertible matrices that forms a basis for $M_n(D)$ as a $D-$algebra. Then,

(1) $\mathfrak{q}_{M_n(\mathfrak{a})}$ is an ideal of $M_n(D)$,

(2) $\mathfrak{q}_{M_n(\mathfrak{a})} \subseteq M_n(\mathfrak{a})$,

(3) If $A - B \in \mathfrak{q}_{M_n(\mathfrak{a})}$, then $\mathcal{J}_{M_n(\mathfrak{a}),A} = \mathcal{J}_{M_n(\mathfrak{a}),B}$.

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Theorem

Let $\mathfrak{a}$ be an ideal of D. Then we have

$$M_n(q_{\mathfrak{a}}) \subseteq q_{M_n(\mathfrak{a})}.$$

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Equalizing Ideals of $\mathrm{Int}(M_n(D))$

Corollary 1

Let D be an one-dimensional, Noetherian, local domain with finite residue field and $\mathfrak{a}$ be an ideal of D. If $\mathfrak{q}_{\mathfrak{a}} \neq 0$, then the set of distinct ideals of the form $\mathcal{J}_{M_n(\mathfrak{a}),A}$ is finite

Corollary 2

Let D be a Noetherian local one-dimensional domain with finite residue field, which is not unibranched. Then the set of ideal of the form $\mathcal{J}_{M_n(\mathfrak{m}),A}$ is finite, where $\mathfrak{m}$ is the maximal ideal of D.

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Outline

Equalizing Ideals of $\mathrm{Int}(T_n(D))$

Definition

Let D be a domain. Let I be an ideal of If $T_n(D)$. We set

$$\mathfrak{q}_I = \{A \in M_n(D) | f(A) - f(0) \in I, \forall f \in \mathrm{Int}(T_n(D))\}.$$

$\mathfrak{q}_I$ is called the equalizing ideal of I in $\mathrm{Int}(M_n(D))$.

Equalizing Ideals of $\mathrm{Int}(T_n(D))$

Theorem

Let I be an ideal of $T_n(D)$ and $A, B \in T_n(D)$. Then the following statements hold.

(1) q_I is an ideal of $T_n(D)$.

(2) $q_I \subseteq I$.

(3) If $A - B \in q_I$, then $\mathcal{J}_{I,A} = \mathcal{J}_{I,B}$.

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Equalizing Ideals of $\mathrm{Int}(T_n(D))$

Lemma

Let $n > 1$ and $f(X) = B_k X^k + \cdots + B_1 X + B_0 \in \mathrm{Int}(T_n(D))$. Then the following statements hold.

(1) $B_0 \in T_n(D)$.

(2) $(B_1)_{ij} \in D$ for all $1 \leq i \leq j \leq n-1$.

Lemma

Let $\mathfrak{a}$ be an ideal of D. Then we have

$$T_n(q_{\mathfrak{a}}) \subseteq q_{T_n(\mathfrak{a})}.$$

Equalizing Ideals of $\mathrm{Int}(T_n(D))$

Theorem

Let I be an ideal of $T_n(D)$. Then we have

$$q_I = \begin{bmatrix} q_{I_{11}} & I_{12} & \dots & I_{1n} \\ 0 & q_{I_{22}} & \dots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & q_{I_{nn}} \end{bmatrix}.$$

One of the main result of this talk is the following corollary.

Corollary

Let I be a nonzero ideal of ideal of $T_n(D)$. Then we have

$$q_I \neq 0.$$

Some other results:

Corollary 1

Let D be an one-dimensional, Noetherian, local domain with finite residue field and $\mathfrak{a}$ be an ideal of D. If $\mathfrak{q}_{\mathfrak{a}} \neq 0$, then the set of distinct left ideals of the form $\mathcal{J}_{T_n(\mathfrak{a}),A}$ is finite

Corollary 2

Let D be a Noetherian local one-dimensional domain with finite residue field, which is not unibranched. Then the set of left ideal of the form $\mathcal{J}_{T_n(\mathfrak{m}),A}$ is finite, where $\mathfrak{m}$ is the maximal ideal of D.

References

References

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Acknowlegement

Thank you for your attention