Atomicity and Factorizations in Monoid Algebras

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Outline

1. Preliminaries on Monoids

2. Preliminaries on Monoid Algebras

3. Atomicity and Factorizations

4. Weaker Notions of Atomicity

General Notation

General notation adopted here:

- $\mathbb{N} := \{1, 2, 3, \dots\}$,
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$,
- $\mathbb{P}$ denotes the set of primes, and
- $\mathbb{F}_q$ denotes the field of q elements.

Monoids

Definition: A commutative monoid is a pair $(M, *)$, where M is a set and $*$ is a binary operation on M satisfying the following conditions.

- $*$ is associative: $b * (c * d) = (b * c) * d$ for all $b, c, d \in M$;
- $*$ is commutative: $b * c = c * b$ for all $b, c \in M$;
- there exists $e \in M$ such that $e * b = b$ for all $b \in M$.

Today's Conventions:

- We call a commutative monoid $(M, *)$ simply a monoid if it is cancellative: $b * d = c * d$ implies $b = c$ for all $b, c, d \in M$.
- For a monoid $(M, *)$, we write M instead of $(M, *)$.
- Any monoid M is written additively, unless otherwise specified: we denote the binary operation of M by $+$ instead of $*$.

Examples of Monoids:

1. Abelian groups: $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{N}$.
2. $\mathbb{N}_0$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ with the standard addition.
3. $\mathbb{N}$, $\mathbb{Q}_{\geq 1}$, and $\mathbb{Q}_{>0}$ with the standard multiplication.
4. The multiplicative set/monoid R^* of any integral domain R.

More Examples of Monoids

Definition: A subset S of a monoid M is called a submonoid of M if S contains the identity element and is closed under the operation of M.

- Submonoids of $\mathbb{N}_0$ are called numerical monoids.
  - $\langle p, q \rangle := \{mp + nq : m, n \in \mathbb{N}_0\}$ for distinct $p, q \in \mathbb{P}$.
  - $\{0\} \cup \mathbb{N}_{\geq k}$ for every $k \in \mathbb{N}$.
- Submonoids of $\mathbb{Q}_{\geq 0}$ are called Puiseux monoids.
  - $\{0\} \cup \mathbb{Q}_{\geq r}$ for any $r \in \mathbb{R}_{>0}$.
  - $\left\langle \frac{1}{p} : p \in \mathbb{P} \right\rangle$.

Examples:

1. Krull monoids,
2. Monoids of zero-sum sequences,
3. Mori monoids,
4. Lattice monoids,
5. Power monoids.

Totally Ordered Monoids

Definition: Let M be a monoid.

- M is torsion-free if for all $b, c \in M$ and $n \in \mathbb{N}$, the equality $nb = nc$ implies that $b = c$.
- The pair $(M, \preceq)$ is a totally ordered monoid (TOM) if $\preceq$ is a total-order relation on M such that for all $b, c, d \in M$, the inequality $b \prec c$ implies $b + d \prec c + d$.

Examples:

1. Every totally ordered abelian group is a TOM.
2. Every Puiseux monoid under the standard order is a TOM.
3. Every submonoid of $(\mathbb{Z}^2, +)$ under the lexicographical order is a TOM.

Theorem (Levi, 1913)

A commutative monoid M can be turn into a TOM if and only if it is cancellative and torsion-free.

Monoid Algebras

Let R be a commutative ring with identity.

Definition (Monoid Algebra)

For a commutative monoid M, the monoid algebra $R[M]$ of M over R is the commutative ring with identity consisting of all polynomial expressions in an indeterminate x with coefficients in R and exponents in M, with polynomial-like addition and multiplication.

Examples of Monoid Algebras:

1. Polynomial rings: $R[x]$ is the monoid algebra of $\mathbb{N}_0$ over R.
2. A numerical monoid algebra: $R[x^p, x^q]$ is the monoid algebra of the numerical monoid $\langle p, q\rangle$ over R, for any distinct $p, q \in \mathbb{P}$.
3. Laurent polynomial rings: $R[x, \frac{1}{x}]$ is the monoid algebra of $\mathbb{Z}$ over R.

Definition: When G is an (abelian) group, $R[G]$ is often called the group algebra of G over R.

Integral Monoid Algebras

Question: Which monoid algebras are integral domains?

Theorem (well-known)

Let M be a commutative monoid, and let R be a commutative ring with identity. Then the following conditions are equivalent.

- *$R[M]$ is an integral domain.*
- *R is an integral domain, and M can be turn into a TOM.*
- *R is an integral domain, and M is both cancellative and torsion-free.*

Definition: Let $R[M]$ be an integral monoid algebra, and let $(M, \preceq)$ be a TOM. Then every nonzero element $f \in R[M]$ can be written canonically as follows:

$$f = r_1 x^{m_1} + \cdots + r_k x^{m_k},$$

where $r_1 r_2 \cdots r_k \neq 0$ and $m_1 \succ m_2 \succ \cdots \succ m_k$. In this case,

- $\operatorname{supp} f := \{m_1, \dots, m_k\}$ is the support of f, and
- $\deg f := m_1$ is the degree of f.

Group of Units and Dimension

Definition: The invertible elements of a monoid M are often called units, and the group of units of M is denoted by $\mathcal{U}(M)$.

Theorem (well-known)

Let R be an integral domain, and let M be a monoid. Then $R[M]^\times = \{rx^m : r \in R^\times \text{ and } m \in \mathcal{U}(M)\}$.

Definition: Let M be a monoid.

- The group of formal differences of M is called its difference group and is denoted by $\mathcal{G}(M)$.
- The rank of M is the rank of the abelian group $\mathcal{G}(M)$.

Theorem

Let R be an integral domain, and let M be a monoid.

- *(Gilmer-Parker, 1974) If $R[M]$ is integral, then $\dim R[M] \ge 1 + \dim R$.*
- *(Okninnski, 1988) If R is Noetherian, then $\dim R[M] = \dim R + \text{rank}\, M$.*

Atomicity

Definition: Let M be a monoid.

- $a \in M \setminus \mathcal{U}(M)$ is an atom (or an irreducible) if for any $b, c \in M$ the equality $a = b + c$ implies that either $b \in \mathcal{U}(M)$ or $c \in \mathcal{U}(M)$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- $b \in M$ is atomic if either b is a unit or $b = a_1 + \cdots + a_n$ for some $a_1, \dots, a_n \in \mathcal{A}(M)$.
- M is atomic if every element of M is atomic.
- An integral domain is atomic if its multiplicative monoid is atomic.

Examples of atomic monoids and atomic domains:

1. Numerical monoids are atomic.
2. Noetherian domains are atomic.
3. Krull monoids/domains are atomic.
4. $\mathbb{Q}_{\geq 0}$ is a non-atomic monoid.
5. $\mathbb{Z} + x\mathbb{Q}[x]$ is a non-atomic domain.

Ascent of Atomicity to Monoid Algebras

Gilmer's Question (1984):
Is a monoid algebra $R[M]$ necessarily atomic provided that R is an atomic domain and M is an atomic monoid?

Gilmer's question admits the following two natural refined versions.

Question 1:

Question (Anderson-Anderson-Zafrullah, 1990)

Is $R[x]$ atomic provided that R is an atomic domain?

Answer: No (Roitman, 1993).

Question 2:

Question

Given a field F, is $F[M]$ atomic provided that M is an atomic monoid?

Answer: No (Coykendall-G., 2019; G.-Rabinovitz, 2023).

Ascent of Atomicity to Monoid Algebras (over fields)

Theorem (G.-Coykendall, 2019)

For every $p \in \mathbb{P}$, there exists an atomic monoid M of rank at most 2 such that the monoid algebra $\mathbb{F}_p[M]$ is not atomic.

Observations:

- Con 1: The theorem only works for specific fields.
- Con 2: When p is odd, the monoid provided in the corresponding construction has rank 2.
- Con 3: The constructed monoid depends on the given field.

Here is a more recent result resolving the three cons.

Theorem (G.-Rabinovitz, 2023)

There exists an atomic rank-1 monoid M such that the monoid algebra $F[M]$ is not atomic for any field F.

Ascent of Atomicity to Monoid Algebras (over integral domains)

Theorem (Coykendall-G., 2019)

There exists an atomic monoid M (with $\text{rank}\, M = \aleph_0$) such that for each integral domain R, the monoid algebra $R[M]$ is a non-atomic integral domain.

Remarks:

- Pro 1: This theorem works not only for fields, but also for integral domains.
- Pro 2: The monoid M is somehow *universal*: it does not depend on the choice of the integral domain R.
- Con: The monoid M is too big: it has infinite rank.

Here is a more recent result resolving the last con.

Theorem (G.-Rabinovitz, 2024)

There exists an atomic rank-1 monoid M such that the monoid algebra $R[M]$ is not atomic for any integral domain R.

Weaker Notions of Atomicity: Open Question 1

Definition: Let M be a monoid.

- $b \in M$ is almost atomic if there exists an atomic element $c \in M$ such that $b + c$ is atomic.
- M is almost atomic if every element of M is almost atomic.
- An integral domain is almost atomic if its multiplicative monoid is almost atomic.

Remark: Every atomic monoid/domain is almost atomic.

Examples:

1. If $M := \left\langle \frac{1}{p} : p \in \mathbb{P} \right\rangle$, then $M \cup \mathcal{G}(M)_{\geq 1}$ is almost atomic but not atomic.
2. $\mathbb{Z} + \mathbb{Z}x + x^2\mathbb{Q}[x]$ is almost atomic but not atomic.

Open Question 1: Does the property of being almost atomic ascend from a monoid M to its monoid algebras over fields?

Weaker Notions of Atomicity: Open Question 2

Definition: Let M be a monoid.

- $b \in M$ is quasi-atomic if there exists $c \in M$ such that $b + c$ is atomic.
- M is quasi-atomic if every element of M is quasi-atomic.
- An integral domain is quasi-atomic if its multiplicative monoid is quasi-atomic.

Remark: Every almost atomic monoid/domain is quasi-atomic.

Examples:

1. $\mathbb{Z}[1/2]_{\geq 0} \cup \mathbb{Z}[1/3]_{\geq 4/3}$ is quasi-atomic but not almost atomic.
2. $\mathbb{Z} + \mathbb{Z}x + x^2\mathbb{R}[x]$ is quasi-atomic but not almost atomic.

Open Question 2: Does the property of being quasi-atomic ascend from a monoid M to its monoid algebras over fields?

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End of Presentation

THANK YOU!