Minimal Factorizations in Reduced Power Monoids

joint work with S. TRINGALI

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MARIE CURIE ACTIONS

What is a power monoid?

H = (multiplicative) monoid with identity 1_H.
$H^\times$ = group of units or invertible elements of H.

$\mathcal{P}(H) := \{X \subseteq H \colon X \neq \emptyset\}$ is a monoid wrt **setwise multiplication**

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\ y \in Y\}.$$

The submonoids of $\mathcal{P}(H)$ will be generically called **power monoids (PMs)** (of H), and H will be referred to as their **ground monoid**.

Power monoids in the literature

- PMs have been a key object of study in semigroup theory since 1960s.
  - **Tamura and Shafer's isomorphism problem (1967):** *is it true that if $\mathcal{P}(H)$ is isomorphic to $\mathcal{P}(K)$, then H is isomorphic to K?*

    T. Tamura, J. Shafer, Math. Japon. 12, 1967.

    Answered (in the negative) by Mogiljanskaja (1973) with a counterexample in the infinite case, the question is still open for finite monoids.
- PMs are a natural algebraic framework for famous problems in additive NT:
  - **Sárközy's conjecture:** For all but finitely many primes p, the set $\mathcal{R}_p \subseteq \mathbb{F}_p$ of quadratic residues $\bmod p$ is an atom[1] in $\mathcal{P}(H)$, with $H = (\mathbb{F}_p, +)$.

    A. Sárközy, Acta Arith. 115, 2012.
  - **Inverse Goldbach conjecture:** Every set of integers that differ from the set of (positive rational) primes by finitely many elements is an atom in $\mathcal{P}(H)$, with $H = (\mathbb{Z}, +)$.

    C. Elsholtz, Mathematika 48, 2001.
- PMs play a central role in the theory of automata and formal languages.

  J. Almeida, Semigroup Forum 64, 2002.

[1] An atom is a non-unit which is not the sum of two non-units.

Power monoids in the literature

- PMs have a rich and interesting arithmetics, as fristly observed in

  Y. Fan, S. Tringali, J. Algebra 512, 2018.

  Since then, they were further considered from this perspective in:

  A. Antoniou, S. Tringali, Pacific J. Math 312, 2021.

  P. Y. Bienvenu, A. Geroldinger, Israel J. Math., to appear.

  They have been the subject of a CrowdMath project recently held by F. Gotti:

  `https://artofproblemsolving.com/polymath/mitprimes2023`

- Due to their "high non-cancellativity", PMs are a leading example in the *unifying theory of factorization* developed by C. and Tringali, based on the interplay between monoids and preorders. Here, the decomposition considered have *irreducible* factors (not necessarily atoms) and satisfy a notion of *minimality*.

  S. Tringali, J. Algebra 602, 2022.

  L. C., S. Tringali, Israel J. Math., to appear.

  L. C., S. Tringali, J. Algebra 630, 2023.

  L. C., S. Tringali, Ark. Math, to appear.

Power monoids are...wild!

$\mathcal{P}(H)$ (also called the **large PM** of H) is a rather complicated object. It is often useful to focus on **"finitary" power monoids**, still messy but much tamer:

$\mathcal{P}_{\mathrm{fin}}(H) := \{X \in \mathcal{P}(H) \colon |X| < \infty\}$, the **small PM** of H;

$\mathcal{P}_{\mathrm{fin},\times}(H) := \{X \in \mathcal{P}_{\mathrm{fin}}(H) \colon X \cap H^\times \neq \emptyset\}$, the **restricted (small) PM** of H;

$\mathcal{P}_{\mathrm{fin},1}(H) := \{X \in \mathcal{P}_{\mathrm{fin}}(H) \colon 1_H \in X\}$, the **reduced (small) PM** of H.

FACTS: If H is *Dedekind-finite*[2], then:

- $\mathcal{P}_{\mathrm{fin},1}(H)$ and $\mathcal{P}_{\mathrm{fin},\times}(H)$ have the same length sets (relative to factorizations into irreducibles);
- $\mathcal{P}_{\mathrm{fin},\times}(H)$ is a divisor-closed submon of $\mathcal{P}_{\mathrm{fin}}(H)$.

If H is *cancellative*[3], then $\mathcal{P}_{\mathrm{fin}}(H)$ is a divisor-closed submonoid of $\mathcal{P}(H)$.

At least for H Dedekind-finite, there is much about PMs that we can understand from the investigation of $\mathcal{P}_{\mathrm{fin},1}(H)$.

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[2] A monoid H is **Dedekind-finite** if $xy = 1_H$ implies $yx = 1_H$.
[3] H is **cancellative** if $xz \neq zy$ and $zx \neq zy$ for all $x, z, y \in H$ with $x \neq y$.

Recent works on reduced power monoids

- Bienvenu and Geroldinger have recently addressed ideal-theoretic and analytic properties of $\mathcal{P}_{\mathrm{fin},0}(S)$, where S is a **numerical semigroup**[4].

  CONJECTURE: For S and S' numerical monoids, $\mathcal{P}_{\mathrm{fin},0}(S) \cong \mathcal{P}_{\mathrm{fin},0}(S') \Leftrightarrow S = S'$.

  P. Y. BIENVENU, A. GEROLDINGER, Israel J. Math., to appear.

- The conjecture was settled in positive by Tringali and Yan, who proved a more general result:

  For S and S' Puiseux monoids[5], $\mathcal{P}_{\mathrm{fin},0}(S) \cong \mathcal{P}_{\mathrm{fin},0}(S') \Leftrightarrow S \cong S'$.

  S. TRINGALI, W. YAN, Proc. Amer. Math. Soc., to appear.

- Tringali and Yan recently classified the authomorphisms of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N})$, establishing that the only non-trivial one is the involution $X \mapsto \max X - X$

  S. TRINGALI, W. YAN, preprint.

- The **arithmetic** of $\mathcal{P}_{\mathrm{fin},1}(H)$ is the main object of study in [Fan & Tringali 2018] and [Antoniou & Tringali, 2021].

  In our work we add to this line of research.

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[4] A **numerical semigroup** is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite.
[5] A **Puiseux monoid** is a submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Irreducibles, atoms, and quarks

Let M be a (multiplicative) monoid.
An element $x \in M$ is a **[non-]unit-divisor** if x is [not] a divisor of 1_M.

A non-unit divisor $a \in M$ is:

- an **irreducible** (of M) if $a \neq xy$ for all non-unit-divisors $x, y \in M$ that properly divide a;
- an **atom** if $a \neq xy$ for all non-unit-divisors $x, y \in M$;
- a **quark** if it is not properly divided by any non-unit-divisor.

REMARKS:

- atom $\Rightarrow$ irreducible and quark $\Rightarrow$ irreducible.
  The converse implications are not true in general.
- If M is Dedekind-finite, the unit-divisors are exactly the units of M and an atom is a non-unit that cannot be written as a product of two non-units, as classically defined.

Irreducibles, atoms, and quarks in reduced PMs

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\mathrm{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\mathrm{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $X = Y$.

iii. $\mathcal{P}_{\mathrm{fin},1}(H)$ is a reduced[6], Dedekind-finite monoid.

iv. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.

v. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if it is a quark.

vi. A set $X \in \mathcal{P}_{\mathrm{fin},1}(H)$ is irreducible but not an atom if and only if $X = \{1_H, x\}$ for some $x \in H \setminus \{1_H\}$ such that $x^2 = 1_H$ or $x^2 = x$.

[6] A monoid M is **reduced** if $M^\times = \{1_M\}$

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\text{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\text{fin},1}(H)$, then $X \subseteq Y$.

$X \mid Y \iff Y = UXV$ for some $U, V \in \mathcal{P}_{\text{fin},1}(H)$

$X = \{1_H\} X \{1_H\} \subseteq UXV = Y$

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\text{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\text{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $X = Y$.

$\Updownarrow$

$X \mid Y$ & $Y \mid X$

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\text{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\text{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $X = Y$.

iii. $\mathcal{P}_{\text{fin},1}(H)$ is a reduced, Dedekind-finite monoid.

$$XY = \{1_H\} = 1_{\mathcal{P}_{\text{fin},1}(H)} \Rightarrow X \cup Y \subseteq \{1_H\} \Rightarrow X = Y = \{1_H\}$$

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\text{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\text{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $X = Y$.

iii. $\mathcal{P}_{\text{fin},1}(H)$ is a reduced, Dedekind-finite monoid.

iv. X is irreducible in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.

(definition)

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\mathrm{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\mathrm{fin},1}(H)$, then $X \subseteq Y$.
ii. X and Y are associated in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $X = Y$.
iii. $\mathcal{P}_{\mathrm{fin},1}(H)$ is a reduced, Dedekind-finite monoid.
iv. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.
v. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if it is a quark.

Proposition 4.11 (iii) in [Tringali, 2021]

$\Downarrow$

$\boxed{\text{Every 2-element set } \{1_H, x\} \in \mathcal{P}_{\mathrm{fin},1}(H) \text{ is an irreducible of } \mathcal{P}_{\mathrm{fin},1}(H)}$

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\text{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\text{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $X = Y$.

iii. $\mathcal{P}_{\text{fin},1}(H)$ is a reduced, Dedekind-finite monoid.

iv. X is irreducible in $\mathcal{P}_{\text{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.

v. X is irreducible in $\mathcal{P}_{\text{fin},1}(H)$ if and only if it is a quark.

vi. A set $X \in \mathcal{P}_{\text{fin},1}(H)$ is irreducible but not an atom if and only if $X = \{1_H, x\}$ for some $x \in H \setminus \{1_H\}$ such that $x^2 = 1_H$ or $x^2 = x$.

$\Leftarrow$: trivial.

If $X = \{1_H, x\}$ with $x \in H \setminus \{1_H\}$ s.t. $x^2 = 1_H$ or $x^2 = x$, then X is IRREDUCIBLE (= QUARK) but <u>NOT</u> an atom since $X^2 = X$.

Proposition 1

Let H be a monoid and let $X, Y, Z \in \mathcal{P}_{\mathrm{fin},1}(H)$. The following hold:

i. If X divides Y in $\mathcal{P}_{\mathrm{fin},1}(H)$, then $X \subseteq Y$.

ii. X and Y are associated in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $X = Y$.

iii. $\mathcal{P}_{\mathrm{fin},1}(H)$ is a reduced, Dedekind-finite monoid.

iv. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if $\{1_H\} \neq X \neq YZ$ for all $Y, Z \subsetneq X$.

v. X is irreducible in $\mathcal{P}_{\mathrm{fin},1}(H)$ if and only if it is a quark.

vi. A set $X \in \mathcal{P}_{\mathrm{fin},1}(H)$ is irreducible but not an atom if and only if $X = \{1_H, x\}$ for some $x \in H \setminus \{1_H\}$ such that $x^2 = 1_H$ or $x^2 = x$.

$\Longrightarrow$: Let $X := \{1_H, x_1, \ldots, x_n\}$ wt $n \geq 1$ be an IRREDUCIBLE but NOT an atom in $\mathcal{P}_{\mathrm{fin},1}(H)$. Then:

$\{1_H\} \neq X = YZ$ for some $\{1_H\} \neq Y, Z \in \mathcal{P}_{\mathrm{fin},1}(H)$.

$\overset{\text{i.}}{\Longrightarrow} Y, Z \subseteq X \overset{\text{v.}}{\Longrightarrow} Y = Z = X \Longrightarrow X = X^2$

Moreover, $X \subseteq X\{1_H, x_1\} \subseteq X^2 = X \Longrightarrow X = X\{1_H, x_1\}$

$\overset{\text{i.}}{\Longrightarrow} \{1_H, x_1\} \subseteq X \overset{\text{v.}}{\Longrightarrow} \underline{X = \{1_H, x_1\} \wedge X = X^2} \Longrightarrow x_1^2 = 1_H$ or $x_1^2 = x_1$

Factorizations and minimal factorizations

Given a monoid M, let $\mathscr{I}(M)$ be the set of irreducibles of M and $\mathscr{F}(M)$ the free monoid over M[7].

A **factorization** (**into irreducibles**) of an element $x \in M$ is an $\mathscr{I}(M)$-word $\mathfrak{a}$ such that $\pi_M(\mathfrak{a}) = x$, where π_M is the **factorization homomorphism** of M[8].

For two M-words $\mathfrak{a}, \mathfrak{b}$, define $\mathfrak{a} \sqsubseteq_M \mathfrak{b}$ if, up to associatedness of the letters, $\mathfrak{a}$ is a subword of a permutation of $\mathfrak{b}$.

An M-word $\mathfrak{a}$ is **minimal** if there is no M-word $\mathfrak{b}$ with $\mathfrak{b} \sqsubseteq_M \mathfrak{a} \not\sqsubseteq_M \mathfrak{b}$. Two M-words $\mathfrak{a}$ and $\mathfrak{b}$ are **equivalent** if $\mathfrak{a} \sqsubseteq_M \mathfrak{b} \sqsubseteq_M \mathfrak{a}$.

Accordingly we can talk about **minimal factorizations** and **equivalent factorizations** of an element $x \in M$.

Given a monoid H, two $\mathcal{P}_{\mathrm{fin},1}(H)$-words are equivalent if and only if they are a permutation of each other.
Thus, a **minimal factorization** of a set $X \in \mathcal{P}_{\mathrm{fin},1}(H)$ is a non-empty word $A_1 * \cdots * A_n$, with $A_1, \ldots, A_n$ **irreducibles**, such that $X = A_1 \cdots A_n$ and $X \neq A_{\sigma(1)} \cdots A_{\sigma(k)}$ for all $k \in [\![1, n-1]\!]$ and every permutation σ of $[\![1, k]\!]$.

[7] We use $*$ to denote the operation on $\mathscr{F}(M)$

[8] The **factorization homomorphism** π_M is the unique extension of the identity map on M to a monoid homomorphism $\mathscr{F}(M) \to M$.

We say that a monoid M is:

- **factorable** if every non-unit-divisor of M factors as a product of irreducibles;
- **BF** if it is factorable and the set of all factorization lengths[9] of any fixed element is bounded;
- **HF** if it is factorable and the factorizations of any element have all the same length;
- **FF** if it is factorable and each element has finitely many inequivalent factorizations;
- **UF** if it is factorable and any two factorizations of an element are equivalent.

$$\begin{tikzcd} & \mathrm{FF} \arrow[dr, Rightarrow] & & \\ \mathrm{UF} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \mathrm{BF} \arrow[r, Rightarrow] & \text{factorable} \\ & \mathrm{HF} \arrow[ur, Rightarrow] & & \end{tikzcd}$$

[9] The **length** of a factorization α is the length of α as a word in the free monoid over M.

We say that a monoid M is:

- **factorable** if every non-unit-divisor of M factors as a product of irreducibles;
- **BF** if it is factorable and the set of all factorization lengths of any fixed element is bounded;
- **HF** if it is factorable and the factorizations of any element have all the same length;
- **FF** if it is factorable and each element has finitely many inequivalent factorizations;
- **UF** if it is factorable and any two factorizations of an element are equivalent.

Replacing factorizations with minimal factorizations in the above definitions results in the notions of **BmF**, **HmF**, **FmF**, and **UmF monoid**. Note that, if M is a factorable monoid, then every non-unit-divisor $x \in M$ has at least one minimal factorization.

If we shift our perspective to regard atoms (rather than irreducibles) as the "building blocks" of the decompositions of interest, we can modify the above definitions accordingly. This will result in the notions of **atomic factorization**, **atomic** monoid, **BF-atomic** monoid, and so on.

Some arithmetical results for reduced PMs

Minimal factorizations were introduced in [Antoniou & Tringali, 2021] who, however, considered **atoms** as their building blocks. They proved that, for any monoid H

$$\mathcal{P}_{\mathrm{fin},1}(H) \textit{ is atomic if and only if } 1_H \neq x^2 \neq x \textit{ for all } x \in H \setminus \{1_H\}.$$

The next Proposition complements this result:

Proposition 2

The following are equivalent for any monoid H:

a. H is aperiodic[10].

b. $\mathcal{P}_{\mathrm{fin},1}(H)$ is BF-atomic.

c. $\mathcal{P}_{\mathrm{fin},1}(H)$ is BF.

d. $\mathcal{P}_{\mathrm{fin},1}(H)$ is FF-atomic.

e. $\mathcal{P}_{\mathrm{fin},1}(H)$ is FF.

The condition a. highlights that atomic factorizations and, more generally, "unrestricted factorizations" into irreducibles are not the best choice possible when it comes to "highly non-cancellative" monoids. Minimal factorizations allow us to overcome these limitations.

Proposition 3

$\mathcal{P}_{\mathrm{fin},1}(H)$ is FmF for every monoid H.

[10] A monoid is **aperiodic** if every non-identity element generates an infinite submonoid.

UmF-ness in reduced PMs, some definitions

Definition [(Almost-)breakable monoid]

A (multiplicative) semigroup (resp., monoid) S is:

- **almost-breakable** if, for every $x, y \in S$, either $xy \in \{x, y\}$ or $yx \in \{x, y\}$;
- **breakable** if $xy \in \{x, y\}$ for all $x, y \in S$.

REMARKS:

- Every breakable semigroup is almost-breakable. The converse need not be true. For instance, let S be the 3-element magma described by the following table:

	x	y	z
x	x	x	x
y	x	y	x
z	z	z	z

  S is an almost-breakable semigroup that is not breakable, since $yz \notin \{y, z\}$.

- Any almost-breakable monoid H is **idempotent**, i.e., $x^2 = x$ for every $x \in H$. Moreover, H is **Dedekind-finite** and **reduced**: if $xy = 1_H$ for some $x, y \in H$, then $y = xy^2 = xy = 1_H$ and hence $x = y = 1_H$.

UmF-ness in reduced PMs

Definition [Trivial ideal extension]

Let H and K be disjoint semigroups. We define a "joint extension" of the operations of H and K to a binary operation on $H \cup K$ by taking $xy = yx := y$ for all $x \in H$ and $y \in K$. We denote the magma obtained in this way by $H \oslash K$ and call it the **trivial ideal extension of K by H**.

REMARK: $H \oslash K$ is a semigroup and it is a monoid if and only if H is. In this case the identity of $H \oslash K$ is the same as the identity of H.

Theorem 1

The following are equivalent for a monoid H:

a. $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF.

b. $H \setminus H^\times$ is an almost-breakable subsemigroup of H, $H^\times$ has order ≤ 2, $H = H^\times \oslash (H \setminus H^\times)$, and $\mathcal{P}_{\mathrm{fin},1}((H \setminus H^\times) \cup \{1_H\})$ is UmF.

Proof of a. $\Rightarrow$ b.

1) $\mathcal{P}_{\mathrm{fin},1}(H)$ UmF $\Rightarrow$
- $H \setminus H^\times$ is an almost-breakable
- $|H^\times| \le 2$
- $H = H^\times \oslash (H \setminus H^\times)$
- $\mathcal{P}_{\mathrm{fin},1}((H \setminus H^\times) \cup \{1_H\})$ UmF

Lemma 1

$\mathcal{P}_{\mathrm{fin},1}(H)$ UmF $\Rightarrow$ for every $x \in H$, either $x^2 = x$ or $x^2 = 1_H$.

1) Assume $\exists\, u, v \in H^\times \setminus \{1_H\}$ s.t. $u \neq v$.

$\Longrightarrow uv \notin \{1_H, u, v\}$ (otherwise $u = v$, or $u = 1_H$, or $v = 1_H$)

Then, since $u^2 = 1_H$

$$\{1_H, u, v, uv\} = \underbrace{\{1_H, u\}\{1_H, v\}}_{} = \underbrace{\{1_H, u\}\{1_H, uv\}}_{}$$

inequivalent minimal fact.

Proof of a. $\Rightarrow$ b.

2) $\mathcal{P}_{\text{fin},1}(H)$ UmF $\Rightarrow$

- $H \setminus H^\times$ is an almost-breakable
- $|H^\times| \le 2$
- $H = H^\times \oslash (H \setminus H^\times)$
- $\mathcal{P}_{\text{fin},1}((H \setminus H^\times) \cup \{1_H\})$ UmF

Lemma 1

$\mathcal{P}_{\text{fin},1}(H)$ UmF $\Rightarrow$ for every $x \in H$, either $x^2 = x$ or $x^2 = 1_H$.

Lemma 2 [Antoniou & Tringali, 2021]

For every submon. K of H, $\mathcal{P}_{\text{fin},1}(K)$ is a divisor-closed submon. of $\mathcal{P}_{\text{fin},1}(H)$.

2) By 1) $H \setminus H^\times$ is a semigroup (closed under $\cdot$), then $K := (H \setminus H^\times) \cup \{1_H\}$ is a submonoid of H.

$\mathcal{P}_{\text{fin},1}(H)$ UmF $\xRightarrow[\text{LEMMA 2}]{}$ $\mathcal{P}_{\text{fin},1}(K)$ UmF.

Proof of a. $\Rightarrow$ b.

3) $\mathcal{P}_{\text{fin},1}(H)$ UmF $\Rightarrow$ $\left|\begin{array}{l} H \setminus H^\times \text{ is an almost-breakable} \\ |H^\times| \le 2 \\ H = H^\times \oslash (H \setminus H^\times) \\ \mathcal{P}_{\text{fin},1}((H \setminus H^\times) \cup \{1_H\}) \text{ UmF} \end{array}\right.$

3) If $H^\times = \{1_H\}$ there is nothing to prove. So let $H^\times = \{1_H, u\}$ with $u^2 = 1$, and $y \in H \setminus H^\times$.

CLAIM: $uy = y$.

Note that $uy \notin \{1_H, u\}$ (otherwise $y = u$ or $y = 1_H$) and assume $uy \neq y$. Then:

$$\{1_H, u, y, uy\} = \underbrace{\{1_H, u\}\{1_H, y\} = \{1_H, u\}\{1_H, uy\}}_{\text{inequivalent minimal fact.}} \lightning$$

Proof of a. $\Rightarrow$ b.

4) $\mathcal{P}_{\text{fin},1}(H)$ UmF $\Rightarrow$ $\left|\begin{array}{l} H \setminus H^\times \text{ is an almost-breakable} \\ |H^\times| \leq 2 \\ H = H^\times \oslash (H \setminus H^\times) \\ \mathcal{P}_{\text{fin},1}((H \setminus H^\times) \cup \{1_H\}) \text{ UmF} \end{array}\right.$

4) We know from 1) that $H \setminus H^\times$ is a semigroup.
Assume it is not almost-breakable, i.e., $\exists\, x, y \in H \setminus H^\times$ s.t. $xy, yx \notin \{x, y\}$. Then:

- $x^2 = x$ and $y^2 = y$ (so $x \neq y$) by LEMMA 1
- $\{1_H, x, y\}$ is irreducible.

So:

$$\{1_H, x, y, xy\} = \underbrace{\{1_H, x\}\{1_H, y\} = \{1_H, x, y\}\{1_H, y\}}_{\text{inequivalent minimal fact.}}$$

The (unit-)cancellative case

TH. 1

$\mathcal{P}_{\mathrm{fin},1}(H)$ UmF $\iff$ $H \setminus H^\times$ alm.-break.
$|H^\times| \le 2$
$H = H^\times \odot (H \setminus H^\times)$
$\mathcal{P}_{\mathrm{fin},1}((H \setminus H^\times) \cup \{1_H\})$ UmF

Corollary 1

If $H \setminus H^\times$ is not almost-breakable, then $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF if and only if H is either trivial or isomorphic to $(\mathbb{Z}_2, +)$.

Remarks:

- If a monoid H is **unit-cancellative**[11] (so, in particular, if it is **cancellative**), there is no idempotent element of H different from the identity and $H \setminus H^\times$ cannot be an almost-breakable semigroup.
- Corollary 1 extends a previous result in [Antoniou & Tringali, 2021]:

  $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF-atomic if and only if H is trivial.

  Recall, in fact, that $\mathcal{P}_{\mathrm{fin},1}(H)$ is atomic if and only if $1_H \neq x^2 \neq x$ for every non-identity element of H.

[11] A monoid H is **unit-cancellative** if $xu \neq x \neq ux$ for all $u, x \in H$ such that u is a non-unit.

Focus on the almost breakable case

In light of Theorem 1, characterizing the UmF-ness of $\mathcal{P}_{\mathrm{fin},1}(H)$ comes down to the case when H is almost-breakable.

Proposition 4

The ground monoid H being almost-breakable is NOT a sufficient condition for $\mathcal{P}_{\mathrm{fin},1}(H)$ to be UmF.

EXAMPLE:

Let H be the unitization of the 4-element magma defined by this table:

	x	y	z	w
x	x	x	x	x
y	x	y	x	y
z	z	z	z	z
w	z	w	z	w

.

H is an almost-breakable monoid and, since $yz = x$ and $wx = z$,

$$\{1_H, x, y, z, w\} = \underbrace{\{1_H, y\}\{1_H, z\}}_{\{1_H, x, y, z\}}\{1_H, w\} = \{1_H, \underbrace{w\}\{1_H, x\}}_{\{1_H, x, z, w\}}\{1_H, y\}.$$

The commutative case

Proposition 5

If H is a breakable monoid, then $\mathcal{P}_{\text{fin},1}(H)$ is UmF.

Since a commutative monoid is almost-breakable if and only if it is breakable, we obtain a complete characterization of UmF-ness of $\mathcal{P}_{\text{fin},1}(H)$, when H is commutative.

Theorem 2

The following are equivalent for a commutative monoid H:

a. $\mathcal{P}_{\text{fin},1}(H)$ is UmF.

b. $H \setminus H^\times$ is a breakable subsemigroup of H, $H^\times$ has order ≤ 2, and $H = H^\times \oslash (H \setminus H^\times)$.

Proposition 5

If H is a breakable monoid, then $\mathcal{P}_{\mathrm{fin},1}(H)$ is UmF.

$\Updownarrow$

$xy \in \{x,y\} \quad \forall x,y \in H$

Lemma 3

For H almost-breakable, the irreducibles of $\mathcal{P}_{\mathrm{fin},1}(H)$ are all and only the sets $\{1_H, x\}$, with $x \in H \setminus \{1_H\}$.

PROOF of PROP. 5: Let H BREAKABLE and set

$$X := \{1_H, x_1, \dots, x_k\}, \quad k \geq 1, \quad x_i \in H \setminus \{1_H\} \ \forall i$$

Every irreducible divisor of X is of the form

$$\{1_H, x_i\} \text{ for some } i \in \{1, \dots, k\}$$

and a minimal factorization α of X is of the form

$$\alpha = \{1_H, x_{i_1}\} * \dots * \{1_H, x_{i_m}\} \text{ with } i_1, \dots, i_m \in \{1, \dots, k\}$$

NOTE: α always exists! $\mathcal{P}_{\mathrm{fin},1}(H)$ is always FmF (so factorable)

Set $I := \{i_1, \dots, i_m\}$.

CLAIM 1 : $m \geq k$. If $m < k$, $\exists\, i \in \{1, \dots, k\} \setminus I$ s.t.

$$x_i = x_{j_1} \cdots x_{j_r} \quad \text{for } r \geq 2 \text{ and } j_1, \dots, j_r \in I$$

But this is IMPOSSIBLE : H breakable $\Rightarrow x_{j_1} \cdots x_{j_r} \in \{x_{j_1}, \dots, x_{j_r}\}$

CLAIM 2 : $x_{i_s} \neq x_{i_t}$ $\forall\, s \neq t$

($\Rightarrow m = k$ and $I = \{1, \dots, k\}$ concluding the proof)

Let $s, t \in \{1, \dots, m\}$ with $s < t$ s.t. $x_{i_s} = x_{i_t}$. Then :

$\{1_H, x_{i_t}\}$

$$\alpha = \underbrace{\{1_H, x_{i_1}\} * \dots * \{1_H, x_{i_{s-1}}\}}_{=\,\zeta_1} * \{1_H, x_{i_s}\} * \underbrace{\{1_H, x_{i_{s+1}}\} * \dots * \{1_H, x_{i_m}\}}_{=\,\zeta_2}$$

Then, for $A := \pi(\zeta_1)$ and $B := \pi(\zeta_2)$:

$$X = \pi(\alpha) = A\{1_H, x_{i_s}\}B = AB \cup A x_{i_s} B = AB = \pi(\zeta_1 * \zeta_2).$$

(α is NOT minimal)

THANK YOU FOR YOUR ATTENTION!