Cross numbers in factorization theory

Aqsa Bashir

University of Graz
Ring Theory Seminar

...ongoing work with Wolfgang Schmid

January 25, 2024

Outline

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- $\mathcal{A}(G)$ the set of all minimal zero-sum sequences over G, and
- $\mathcal{A}^*(G)$ the set of all non-empty zero-sum free sequences over G.

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- Note that $\frac{1}{\exp(G)} + \mathsf{k}(G) \le \mathsf{K}(G)$.

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- Cross numbers of certain zero-sum sequences are invariants which describe the arithmetic of Krull monoids.

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• The precise value of $\mathsf{K}(G)$ is not known in general and in addition no counterexample is known so far for which the inequality is strict.

Remarks-II

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  2. Let $G = C_2^2 \oplus G_p$ where G_p is a p-group for some odd prime p. Then $\mathsf{K}^*(G) = \mathsf{K}(G)$.

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For a finite group G, any sequence S of $|G|$ elements contain a zero-sum subsequence S' with $\mathsf{k}(S') \leq 1$.

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This was proved for **abelian groups** by Geroldinger (1991) and by Elledge-Hurlbert (2005) via graph pebblings. The inverse problem was recently studied by Zhong (2021).

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  ** An atomic monoid is factorial if and only if it is half-factorial and length-factorial.

Lemma (Skula - Zaks, 1976)

Let $G_0 \subset G$ be a non-empty subset. Then TFAE.

- *G_0 is half-factorial.*
- $\mathsf{k}(S) = 1$ *for all* $S \in \mathcal{A}(G_0)$.
- $\mathsf{L}(S) = \{\mathsf{k}(S)\}$ *for all* $S \in \mathcal{A}(G_0)$.

Lemma

Let $G_0 \subset G$ *be a non-empty subset.*

• *If there is an* $S \in \mathcal{A}(G_0)$ *with* $\mathsf{k}(S) = 1$ *and* $|\operatorname{supp}(S)| \geq 2$ *then* G_0 *is not length-factorial.*

• *If there are distinct* $S_1, S_2 \in \mathcal{A}(G_0)$ *with* $\mathsf{k}(S_1) \neq 1$ *and* $\mathsf{k}(S_2) \neq 1$, *then* G_0 *is not length-factorial.*

Theorem (Characterization of length-factoriality)

Let $G_0 \subset G$ such that $\langle G_0 \rangle = G$ and $g \in \langle G_0 \setminus \{g\} \rangle$ for all $g \in G_0$. TFAE

a. *G_0 is length-factorial but not factorial.*

b. *$G_0 \setminus \{0\} = \{e_1, \ldots, e_r\}$ where $(e_1, \ldots, e_r)$ is a basis of G, $r \in \mathbb{N}$ and $\operatorname{ord}(e_i) = n \geq 2$ for all $i \in [1, r]$ and $r + 1 \neq n$.*

In particular, $G \cong C_n^r$.

Outline

A list of some sets of invariants

1. $\mathcal{L}(H) = \{\mathsf{L}(a) : a \in H\}$ the *system of sets of lengths*,
2. $\Delta(\mathcal{L}) = \bigcup_{\mathsf{L} \in \mathcal{L}} \Delta(\mathsf{L}) \subset \mathbb{N}$ the *set of distances of* $\mathcal{L}$,
3. $\mathcal{R}(\mathcal{L}) = \{\rho(\mathsf{L}) : L \in \mathcal{L}\} \subset \mathbb{Q}_{\ge 1}$ the *set of elasticities of* $\mathcal{L}$,
4. $\mathsf{Ca}(H) = \{\mathsf{c}(a) : a \in H \text{ with } \mathsf{c}(a) > 0\} \subset \mathbb{N}$ the *set of catenary degrees*, ...

Notation

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$$\mathsf{w}(G) \subseteq \frac{1}{\exp(G)}[1, \exp(G)\mathsf{k}(G)]\,.$$

Cross numbers up to 1 and beyond

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     (e.g., $\mathsf{W}(C_{2p^k}) = \frac{2}{2p^k}[1, 3p^k - 1]$,
     $\mathsf{W}(C_{3p}) = \frac{1}{3p}\{[2, 6p - p - 2] \setminus \{6p - p - 3\}\}$,
     $\mathsf{W}(C_{5\times 7}) = \frac{1}{35}\{[2, 59] \setminus \{56, 57, 58\}\}$.......).

Lemma

Let G be a finite abelian group. We have

$$\frac{1}{\exp(G)}[1, \exp(G)-1] \subseteq \mathsf{w}(G)\,.$$

Our results

Theorem (1)

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ *be a finite abelian p-group with* $1 = n_0 < n_1 \mid \ldots \mid n_r$.

1. *Suppose that* p *is either odd or that* $p = 2$ *with* $n_{r-1} = n_r$. *Then*
   $\mathsf{W}(G) = \frac{1}{\exp(G)}[2, \exp(G)\mathsf{K}(G)]$ *and*
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   $\mathsf{W}(G) = \frac{1}{\exp(G)}[2, \exp(G)\mathsf{K}(G)]$ *and*
   $\mathsf{w}(G) = \frac{1}{\exp(G)}[1, \exp(G)\mathsf{k}(G)]$.
2. *Suppose* $p = 2$ *and* $n_{r-1} < n_r$. *Then*
   $\mathsf{W}(G) = \frac{2}{\exp(G)}[1, \frac{\exp(G)}{2}\mathsf{K}(G)]$ *and*
   $\mathsf{w}(G) = \frac{1}{\exp(G)}[1, \exp(G)\mathsf{k}(G)]$.

Theorem (2)

Let $G = C_{2p^k}$ with $k \in \mathbb{N}$ and let p be a prime. Then

$$\mathsf{W}(C_{2p^k}) = \frac{2}{\exp(G)}[1, \frac{\exp(G)}{2}\mathsf{K}(G)]$$

and

$$\mathsf{w}(C_{2p^k}) = \frac{1}{\exp(G)}[1, \exp(G)\mathsf{k}(G)] .$$

Theorem (3)

1. *Let* $G = G_1 \oplus \ldots \oplus G_s$*, where* $s \in \mathbb{N}$, $\exp(G_i) = n$ *for all* $i \in [1, s]$*. Then*

$$\frac{1}{n}[1, (n-1)s] \subseteq \mathsf{w}(G)\,.$$

2. *Let* G *be a finite abelian group with* $\exp(G) = n \geq 2$*. Then there exist constants* $c, s^* \in \mathbb{N}$ *such that, for all* $s \geq s^*$,

$$[1, s\exp(G)\mathsf{k}(G) - c] \subseteq \exp(G)\mathsf{w}(G^s)\,.$$

In particular, if $\mathsf{k}(G^s) = s\mathsf{k}(G)$*, then* $\exp(G)\mathsf{w}(G^s)$ *is an interval, apart from a globally bounded upper part.*

The second statement states that, in groups where the rank is large with respect to the exponent, $\exp(G)\mathsf{w}(G)$ is an interval, apart from a globally bounded upper part.

The second statement states that, in groups where the rank is large with respect to the exponent, $\exp(G)\mathsf{w}(G)$ is an interval, apart from a globally bounded upper part. More precisely, there is a global constant $c \in \mathbb{N}$ such that $\exp(G)\mathsf{w}(G) \cap [1, \max \mathsf{w}(G) - c] = [1, \max \mathsf{w}(G) - c]$.

Proposition (1)

Let G and G' be finite abelian groups.

1. *Let* $|G|, |G'| \geq 3$. *If* $\mathsf{W}(G) = \mathsf{W}(G')$ *then* $\exp(G) = \exp(G')$ *and* $\mathsf{K}(G) = \mathsf{K}(G')$.
2. *Let G be a p-group for some odd prime p. We have* $\mathsf{W}(G) = \mathsf{W}(G')$ *if and only if* $\exp(G) = \exp(G')$ *and* $\mathsf{K}(G) = \mathsf{K}(G')$.
3. *Let G be a finite abelian p-group for some prime p. We have* $\mathsf{w}(G) = \mathsf{w}(G')$ *if and only if* $\exp(G) = \exp(G')$ *and* $\mathsf{k}(G) = \mathsf{k}(G')$.

Note: If $|G|, |G'| \in \{1, 2\}$, then $\mathsf{W}(G) = \mathsf{W}(G') = \{1\}$ but $\exp(G), \exp(G') \in \{0, 2\}$.

Remarks-III

• Let $p = 2$ and $G_p = C_{p^{k_1}} \oplus \ldots \oplus C_{p^{k_r}}$ with $1 \leq k_1 \leq \ldots \leq k_r$. Let $G = G_p$ with $k_{r-1} = k_r$ and $G' = G_p$ with $k_{r-1} \neq k_r$. Then $\exp(G) = \exp(G')$ and for suitable choices of r and k_i's it is possible to have $\mathsf{K}(G) = \mathsf{K}(G')$ too (indeed, for instance take $G = C_4^3$ and $G' = C_2^3 \oplus C_4$ then $\exp(G) = \exp(G') = 4$ and $\mathsf{K}(G) = \mathsf{K}(G') = \frac{10}{4}$) but in any case, Theorem (1) tells that $\mathsf{W}(G) \neq \mathsf{W}(G')$. Thus the Proposition (1).2 only works for odd primes.

Question: For some groups G, G', does $\exp(G) = \exp(G')$, $\mathsf{K}(G) = \mathsf{K}(G')$ and $\mathsf{W}(G) = \mathsf{W}(G')$ implies G, G' have same ranks?

Question: For some groups G, G', does $\exp(G) = \exp(G')$, $\mathsf{K}(G) = \mathsf{K}(G')$ and $\mathsf{W}(G) = \mathsf{W}(G')$ implies G, G' have same ranks?
NO!

Question: For some groups G, G', does $\exp(G) = \exp(G')$, $\mathsf{K}(G) = \mathsf{K}(G')$ and $\mathsf{W}(G) = \mathsf{W}(G')$ implies G, G' have same ranks?
NO!
For instance, if $G = C_3^4 \oplus C_9$ and $G' = C_9^4$ then $\mathsf{W}(G) = \mathsf{W}(G')$ by Theorem (1) but G and G' have different rank. Therefore, $\exp(G) = \exp(G')$, $\mathsf{K}(G) = \mathsf{K}(G')$ and $\mathsf{W}(G) = \mathsf{W}(G')$ does not give any relation between rank of G and the rank of G'.

Proposition (2)

Let p, q be odd primes, then $\mathrm{W}(C_{pq}) = \frac{1}{pq} + \mathrm{w}(C_{pq})$

Note that their is no direct connection between W(G) and w(G) for many other groups. For instance, Note that their is no direct connection between W(G) and w(G) for many other groups. For instance,

1. if $G \cong C_{2^k}$ for some $k \in \mathbb{N}$ then by Theorem (1) we have $\mathsf{W}(G) = \frac{2}{2^k}[1, 2^{k-1}]$ and $\mathsf{w}(G) = \frac{1}{2^k}[1, 2^k - 1]$ Now, for some $l = \frac{\lambda}{2^k} \in \mathsf{w}(C_{2^k})$ with λ even, clearly $l + \frac{1}{2^k} \notin \mathsf{W}(C_{2^k})$.

Note that their is no direct connection between $\mathsf{W}(G)$ and $\mathsf{w}(G)$ for many other groups. For instance,

1. if $G \cong C_{2^k}$ for some $k \in \mathbb{N}$ then by Theorem (1) we have $\mathsf{W}(G) = \frac{2}{2^k}[1, 2^{k-1}]$ and $\mathsf{w}(G) = \frac{1}{2^k}[1, 2^k - 1]$ Now, for some $l = \frac{\lambda}{2^k} \in \mathsf{w}(C_{2^k})$ with λ even, clearly $l + \frac{1}{2^k} \notin \mathsf{W}(C_{2^k})$.
2. if $G \cong C_{2p^k}$ for some odd prime p and some $k \in \mathbb{N}$. Then by Theorem (2) $\mathsf{W}(G) = \frac{2}{2p^k}[1, \frac{3p^k - 1}{2}]$ and $\mathsf{w}(G) = \frac{1}{2p^k}[1, 3p^k - 2]$. Thus for some $\frac{\lambda}{2p^k} \in \mathsf{w}(G)$ with λ even, $\frac{\lambda+1}{2p^k}$ is not a cross number of any minimal zero-sum sequence over C_{2p^k}. Another message of this example is that both primes p and q being odd is a necessary condition in above proposition.

Structure of W(G) and w(G)

- Are $\exp(G)\mathrm{W}(G)$ and $\exp(G)\mathrm{w}(G)$ intervals? If not, is there a visible gap structure?

Thank you for your attention!